604.03

CITY UNIVERSITY

London

BSc Degrees in Mathematical Science
Mathematical Science with Statistics
Mathematical Science with Computer Science
Mathematical Science with Finance and Economics
MMath Degrees in Mathematical Science

PART III EXAMINATION

Dynamical Systems

xxth of January 2010 14:30 am – 16:30 am

Time allowed: 2 hours

Full marks may be obtained for correct answers to THREE of the FIVE questions.

If more than THREE questions are answered, the best THREE marks will be credited.

Turn over . . .

1. Consider the following second order differential equation

$$\ddot{x} + \dot{x} + \mu x^3 + \nu \dot{x}^5 = 0 \qquad \mu \in \mathbb{R}^+, \nu \in \mathbb{R}.$$

(*i*) Find a suitable variable transformation which changes the second order differential equation into a system of two first order differential equations.

(*ii*) Determine the fixed point of the system. State the linearization theorem and decide whether it can be used to draw conclusions from it with regard to the stability of the fixed point.

(*iii*) State the Lyapunov stability theorem. Take the function

$$V(x_1, x_2) = \alpha x_1^4 + 2x_2^2, \qquad \alpha \in \mathbb{R}^+$$

as a candidate for a Lyapunov function. Find a relation between α and μ and a range for ν, such that $V(x_1, x_2)$ becomes a weak Lyapunov function for the dynamical system constructed in (*i*).

(*iv*) For $\alpha = \mu$ and $\nu = 0$ decide whether any of the two functions

$$V_1(x_1, x_2) = x_1^2 + x_2^2 \qquad \text{and} \qquad V_2(x_1, x_2) = x_1^3 + x_2^3$$

is a Lyapunov function for the above system.

(*v*) State the extension of the Lyapunov stability theorem and conclude from it that the fixed point is asymptotically stable for the values of α, μ and ν found in (*iii*).

Turn over ...

2. Consider the dynamical system of the form

$$\begin{aligned}\dot{x}_1 &= x_2 + x_1(x_1^2 + x_2^2 - 5)(1 - x_1^4 - x_2^4 - 2x_1^2 x_2^2)\\ \dot{x}_2 &= -x_1 + x_2(x_1^2 + x_2^2 - 5)(1 - x_1^4 - x_2^4 - 2x_1^2 x_2^2)\end{aligned}$$

(i) Change the variables of the system to polar coordinates, using the conventions $x_1 = r\cos\vartheta$ and $x_2 = r\sin\vartheta$. Deduce that the origin is the only fixed point of the system.

(ii) Use the Poincaré-Bendixson theorem to argue that there is at least one limit cycle in the annular region

$$\mathcal{D} = \{(r,\vartheta) : 2 \leq r \leq 3\}\ .$$

(iii) Sketch the phase portrait for the above system.

(iv) State the defininiton for α and ω limit sets and subsequently compute all of them. State the definition of a limit cycle using the notion of α and ω limit sets. Compute the limit cycles explicitly and decide whether they are stable, unstable or semistable. Compare your results with your answer to (ii).

(v) State Bendixson's criterium and decide whether it can be used to argue that the domain

$$\hat{\mathcal{D}} = \{(r,\vartheta) : 3 \leq r \leq 5\}$$

contains a limit cycle.

Turn over ...

3. (*i*) Describe what is meant by bifurcation theory, a bifurcation and a bifurcation diagram. Consider the one dimensional system

$$\dot{x} = x^3 + \gamma x^2 - \lambda x \quad \text{with } \gamma, \lambda \in \mathbb{R}.$$

Find the fixed points for the system. For $\gamma = 0$ find the pitchfork deformation point for the above system. Show that when $\gamma \neq 0$ this point transforms into a transcritical bifurcation point. Find the turning point of the system as a function of γ.

(*ii*) Consider the dynamical system

$$\begin{aligned} \dot{x}_1 &= 9x_2 + 3x_1^2 \\ \dot{x}_2 &= \lambda x_2 - 2x_1^2 x_2 - 9x_1 - 2x_1^3 + \alpha x_1^2 \end{aligned}$$

with $\lambda \in \mathbb{R}$ being a bifurcation parameter and constant $\alpha \in \mathbb{R}$. For which values of α is the origin an asymptotically stable fixed point? Hint: You may use the stability index

$$\begin{aligned} I &= \omega\left(Y_{111}^1 + Y_{122}^1 + Y_{112}^2 + Y_{222}^2\right) + Y_{11}^1 Y_{11}^2 - Y_{11}^1 Y_{12}^1 + Y_{11}^2 Y_{12}^2 \\ &\quad + Y_{22}^2 Y_{12}^2 - Y_{22}^1 Y_{12}^1 - Y_{22}^1 Y_{22}^2 \end{aligned}$$

where

$$Y_{jk}^i = \left.\frac{\partial^2 Y_i}{\partial y_j \partial y_k}\right|_{(0,0)} \qquad \text{and} \qquad Y_{jkl}^i = \left.\frac{\partial^3 Y_i}{\partial y_j \partial y_k \partial y_l}\right|_{(0,0)}.$$

(*iii*) State the Hopf bifurcation theorem and use it to decide whether the system in (*ii*) possesses a Hopf bifurcation point for $\lambda = 0$ when $\alpha = 2$ and $\alpha = 4$.

Turn over ...

4. (*i*) What is meant by a Hamiltonian system in two dimensions, equations of motion and an autonomous system?

(*ii*) Provide a criterium that can be used to decide whether a system is Hamiltonian or not. Employ the criterium to determine the value of μ such that the system

$$\begin{aligned}\dot{x}_1 &= 3x_1^2x_2^2 + 2x_1 + 5x_2\\ \dot{x}_2 &= -\mu x_1^2x_2^3 - 2x_2 + \sin(x_1^5)\end{aligned}$$

becomes a Hamiltonian system.

(*iii*) Provide a definition for a potential system. Derive the Hamiltonian function for the dynamical system

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -2x_1 + \frac{20x_1}{1+x_1^2}\end{aligned}$$

and confirm that it is a potential system. Fix the potential in such way that its value at zero is zero.

(*iv*) Compute all fixed points of the system in (*iii*) and determine their nature by making use of the fact that the system is a potential system.

(*v*) Sketch the potential for the system in (*iii*) and compute the equation for the separatrices by making use of the fact that the Hamiltonian is conserved along a trajectory. Draw the phase portrait for the system. Include the separatrices and some representative trajectories. Include the direction of time on the trajectories and provide a reasoning for your choices. Explain in which areas of your diagram the motion is bounded.

Turn over . . .

5. Consider the following difference equation

$$x_{n+1} = F(x_n) = 8\lambda x_n - 4\lambda x_n^2 \qquad \text{for } \lambda \in \mathbb{R}^+.$$

λ is taken to be the bifurcation parameter.

(*i*) Depending on the values of λ, determine the nature of the fixed points and their stability.

(*ii*) State the condition which determines the existence of a 2-cycle. Show that 2-cycles for the above system are governed by the solutions of the equation

$$1 + 8\lambda - 4x\lambda - 32x\lambda^2 + 16x^2\lambda^2 = 0 \ .$$

Compute the solution of this equation and use it to argue that the existence of a 2-cycle requires $\lambda \geq 3/8$.

(*iii*) Determine the domain of stability for the 2-cycle.

Internal Examiner: Professor A. Fring
External Examiners: Professor J. Billingham
Professor E. Corrigan