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CITY UNIVERSITY

London

BSc Degrees in Mathematical Science
Mathematical Science with Statistics
Mathematical Science with Computer Science
Mathematical Science with Finance and Economics
MMath Degrees in Mathematical Science

Part III Examination

Dynamical Systems

6th of January 2011 10:00 am – 12:00 am

Time allowed: 2 hours

Full marks may be obtained for correct answers to THREE of the FIVE questions.

If more than THREE questions are answered, the best THREE marks will be credited.

Turn over ...

1. (*i*) Provide a definition for a simple linear system.

(*ii*) For a two dimensional dynamical system prove the following statement: The only fixed point of a simple linear system is the origin.

(*iii*) Consider the linear dynamical system of the form

$$\begin{aligned} \dot{x}_1 &= ax_1 + bx_2 \\ \dot{x}_2 &= cx_1 + dx_2 \end{aligned} \qquad \text{with} \;\; a, b, c, d \in \mathbb{R}.$$

Assume that the Jacobian matrix can be brought into the Jordan normal form

$$J = \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix} \qquad \text{with} \;\; \lambda_+, \lambda_- \in \mathbb{R}.$$

Prove that for $\lambda_+ > \lambda_- > 0$ the origin is an unstable node, whereas for $\lambda_- < \lambda_+ < 0$ the origin is a stable node.

Turn over . . .

2. Consider the dynamical system of the form

$$\begin{aligned}\dot{x}_1 &= x_2 + x_1(2 - x_1^6 - x_2^6 - 3x_1^4x_2^2 - 3x_1^2x_2^4)(x_1^2 + x_2^2 - 6)\\ \dot{x}_2 &= -x_1 + x_2(2 - x_1^6 - x_2^6 - 3x_1^4x_2^2 - 3x_1^2x_2^4)(x_1^2 + x_2^2 - 6)\end{aligned}$$

(i) Change the variables of the system to polar coordinates, using the conventions $x_1 = r\cos\vartheta$ and $x_2 = r\sin\vartheta$. Deduce that the origin is the only fixed point of the system.

(ii) Identify the limit cycles of the system and sketch its phase portrait.

(iii) State the Poincaré-Bendixson theorem and decide whether one can use it to argue that there is at least one limit cycle in the annular region

$$\mathcal{D} = \{(r, \vartheta) : 1 \leq r \leq 3\} \ .$$

(iv) State the defininiton for α and ω limit sets and subsequently compute all of them. State the definition of a limit cycle using the notion of α and ω limit sets. Compute the limit cycles explicitly and decide whether they are stable, unstable or semistable.

(v) State Bendixson's criterium and decide whether it can be used to argue that the domain

$$\hat{\mathcal{D}} = \left\{(r, \vartheta) : 2 \leq r \leq \frac{5}{2}\right\}$$

contains a limit cycle.

Turn over ...

3. (*i*) Describe what is meant by a bifurcation. For a one dimensional dynamical system provide a precise definition of a) a turning point b) a transcritical bifurcation and c) a pitchfork bifurcation.

(*ii*) Consider the van der Pol differential equation

$$\ddot{x} + \lambda(x^2 - 1)\dot{x} + x = 0.$$

Find a suitable transformation to convert this equation into a two dimensional dynamical system. Use the linearization theorem to establish that this system possesses a Hopf bifurcation

(*iii*) State the Hopf bifurcation theorem and decide whether it can be applied using the stability index to confirm the result from (*ii*).

Hint: The stability index I is defined as

$$\begin{aligned} I &= \omega\left(Y^1_{111} + Y^1_{122} + Y^2_{112} + Y^2_{222}\right) + Y^1_{11}Y^2_{11} - Y^1_{11}Y^1_{12} + Y^2_{11}Y^2_{12} \\ &\quad + Y^2_{22}Y^2_{12} - Y^1_{22}Y^1_{12} - Y^1_{22}Y^2_{22} \end{aligned}$$

where

$$Y^i_{jk} = \left.\frac{\partial^2 Y_i}{\partial y_j \partial y_k}\right|_{(0,0)} \qquad \text{and} \qquad Y^i_{jkl} = \left.\frac{\partial^3 Y_i}{\partial y_j \partial y_k \partial y_l}\right|_{(0,0)}.$$

Turn over . . .

4. (a) Consider the harmonic oscillator in two dimensions described by the Hamiltonian

$$H(x_1, x_2, p_1, p_2) = \frac{1}{2m}(p_1^2 + p_2^2) + \frac{k}{2}(x_1^2 + x_2^2) \qquad \text{with } \ k, m \in \mathbb{R}.$$

(*i*) Derive the equations of motion for this Hamiltionian.

(*ii*) Use the definition for the Poisson bracket to show that the two quantities

$$\begin{aligned} L(x_1, x_2, p_1, p_2) &= x_1 p_2 - x_2 p_1 \\ K(x_1, x_2, p_1, p_2) &= \frac{1}{2m}(p_1^2 - p_2^2) + \frac{k}{2}(x_1^2 - x_2^2) \end{aligned}$$

are conserved in time, i.e. $\dot{L} = \dot{K} = 0$.

(*iii*) State the Jacobi-Poisson theorem and employ it to construct a new conserved quantity from L and K. Verify explicitly that this new quantity is indeed conserved.

(*iv*) Show that the conserved quantity constructed in *iii*) is not independent of the previous ones.

Turn over ...

5. Consider the following difference equation

$$x_{n+1} = F(x_n) = x_n^2 - 4\lambda x_n - 12\lambda^2 \qquad \text{for } \lambda \in \mathbb{R}^+.$$

λ is taken to be the bifurcation parameter.

(i) Depending on the values of λ, determine the nature of the fixed points and their stability.

(ii) State the condition which determines the existence of a 2-cycle. Show that 2-cycles for the above system are governed by the solutions of the equation

$$x^2 - 4\lambda x + x - 12\lambda^2 - 4\lambda + 1 = 0 \ .$$

Compute the solution of this equation and find the regime in which a 2-cycle exists.

(iii) Determine the domain of stability for the 2-cycle.

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