Dynamical Systems

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ABSTRACT: This is an introduction to dynamical systems.

1. Introduction

1.1 The notion of a dynamical system

Let us start by assembling some general notions and concepts to make the subject we are dealing with precise.

Definition: *A dynamical system describes the evolution of some quantities* $x_1, x_2, \ldots, x_n$, *whose meaning will be specified later, of a physical, chemical, biological, economical etc. system as a function of time* t *or a similar variable (temperature* T*, number density* $\rho, \ldots$*)*
For the time being we take the variable to be usually t, keeping however in mind that it could be more general or something else. The central question for a dynamical system is then to determine how a particular state of the system evolves in time subject to some specific rules and what kind of solutions these rules allow. More precisely:

Definition: *The state of a system is described by a collection of continuous (or discrete) parameters at a particular time, say* t_0*, as* $x_1(t_0), x_2(t_0), \ldots, x_n(t_0)$.

Definition: *The space* $\chi(t, \vec{x})$ *of all possible states is a subspace of the Euclidean space, called the phase space (or state space).*

Definition: *The law of evolution (or equation of motion) in time can predict* $\chi(t, \vec{x})$*, if we know some initial state* $\chi(t_0, \vec{x}_0)$.

Mathematically the law of evolution can be algebraic, functional, a differential equation, an integral equation or a mixture of them. In general terms the law of evolution is a map from the phase space into itself $\pi(t_1, t_0) : \chi \to \chi$ evolving the system from some time t_0 to another time $t_1$obeying the properties for two consecutive compositions

$$\pi(t_2, t_1) \circ \pi(t_1, t_0) = \pi(t_2, t_0), \tag{1.1}$$

$$\pi(t_0, t_0) = \mathbb{I}. \tag{1.2}$$

The first equation (1.1) indicates that if we use the law of evolution to evolve the system from some time t_0 to another time t_1 and subsequently from t_1 to t_2, the direct evolution from t_0 to t_2 gives the same result. The second equation (1.2) states that when we evolve from t_0 to the same time t_0 the map simply reduces to the identity map $\mathbb{I}$. Mostly one also assumes that the evolution map only depends on the difference of the initial and final time, denoted as $t := t_1 - t_0$, such that one can abbreviate $\pi(t_1, t_0) =: \pi_t$. With this assumption and the specified notations the equations (1.1) and (1.2) simplify to

$$\pi_s \circ \pi_t = \pi_{s+t}, \tag{1.3}$$

$$\pi_0 = \mathbb{I} \; . \tag{1.4}$$

Having now a rough general idea of what dynamical systems are, we briefly comment on how the manner this subject has developed over the years.

1.2 Historical Remarks

The beginning of the subject can be traced back to ancient astronomy, when people started to try to "explain" the motion of planets. The first qualitative investigation were carried out in the middle ages.

In the 17-th century Kepler and Galilei brought the subject forward by their observations. At the end of the 17-th century Newtonian mechanics provided the theoretical explanation of Kepler's and Galilei's observations. The first analytical methods were developed thereafter by Euler, Lagrange, Laplace, Hamilton and Jacobi. The end of the 18-th and the entire 19-th century was dominated by trying to solve the three-body problem. Bruns and Poincaré found that the three-body problem can not be solved with standard methods, which led to a crisis of the entire subject. The contribution of Lyapunov was to introduce a new point of view. Instead of seeking explicit solutions for a dynamical system, he developed the theory of stability. In 1927 Birkhoff developed topological dynamics. In 1937 Andronov and Pontryagin invented structural stability. The so-called Lefschetz School was concerned with the study of the nonlinear oscillator. Smale invented differentiable dynamical systems. Further important contributions were made by Hopf, Hedlund, Sinai, Bowen, Thom, Ruelle and many more authors. Only as late as 1960 the crisis concerned with the three-body problem was overcome, when Kolmegorov, Arnold and Moser showed that there exist quasi-periodic solutions to the n-body problem, which guarantee the stability of the solar system.

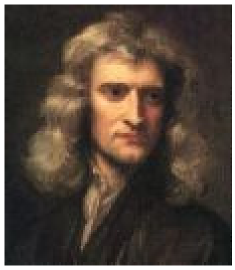

Figure 1: Newton

2. Linear systems, preliminary notions

2.1 Solutions, Phase portraits, fixed points, qualitative behaviour

In general we consider dynamical systems of the form

$$\boxed{\dot{\vec{x}} = \vec{F}(\vec{x})}. \tag{2.1}$$

Here the dot denotes a time derivative, i.e. $\dot{x} = dx/dt$, $\vec{F}$ is a map (a vectorfield) $\mathbb{R}^n \to \mathbb{R}^n$ and $\vec{x}$ is an n-dimensional vector, i.e. $\vec{x} = (x_1, x_2, \ldots, x_n)$. (In this course we will not make a pendantic distinction between column and row vectors.) Alternatively we can write equation (2.1) in components as

$$\dot{x}_i = F_i(\vec{x}) = F_i(x_1, x_2, \dots, x_n) \qquad \text{for } 1 \le i \le n. \tag{2.2}$$

We are particularly interested in the case $n = 2$ (see section 3 of the course) and restrict very frequently our discussion to that particular case.

Definition: *A solution for the system (2.1) is a set of functions $\{x_1(t), x_2(t), \dots, x_n(t)\}$ satisfying (2.1) for a given vectorfield $\vec{F}$.*

The solution can be depicted in a *phase plane*, see figure 1. The arrow indicates the direction of time. We call a particular line a *trajectory*. A collection of such curves constitutes a *phase portrait*. For many general considerations it is enough to study the *qualitative behaviour*, which means the precise numbers are not relevant in that context. Often they are also not available since the system can not be solved explicitly.

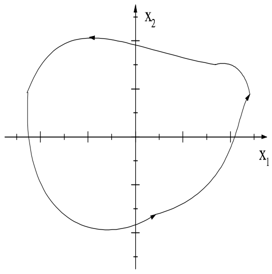

Figure 2: A trajectory in the phase plane.

Definition: *A solution $x(t) = x_f$ of (2.1) which stays the same for all time is called a fixed point.*

For a fixed point we therefore have the relations

$$\dot{\vec{x}}(t) = \vec{x}_f = \vec{F}(\vec{x}_f) = 0. \tag{2.3}$$

Let us consider an easy example to illustrate all the notions and definitions just introduced

$$\dot{x}_1 = -x_1 \qquad \text{and} \qquad \dot{x}_2 = -x_2. \tag{2.4}$$

We may easily solve these equations separately by

$$x_i(t) = k_i \exp(-t) \qquad \text{with } i = 1, 2,\ k_i \in \mathbb{R}. \tag{2.5}$$

The origin is clearly a fixed point which follows directly with properties (2.3). Dividing the two equations in (2.5) we obtain $x_1 = k_1/k_2 x_2$, which means the trajectories are straight lines in the x_1/x_2-plane. At the moment we do not have any information on the direction of time. To obtain this we can study the limit of time going to infinity

$$\lim_{t\to\infty} x_i(t) = \lim_{t\to\infty} k_i \exp(-t) = 0, \tag{2.6}$$

which means the fixed point will be reached in the infinite future. This implies that all time directions point towards the origin. Assembling all this information we obtain the phase portrait as depicted in figure 3.

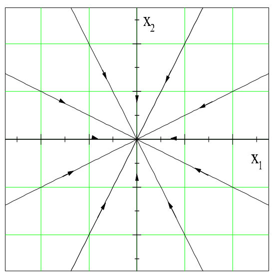

Figure 3: Phase portrait for the system (2.4).

2.2 Linear systems

The main concern of this course will be non-linear systems, but to start with we need to have a good understanding of linear systems, which in many cases serve as a very good benchmark.

Definition: *In case the map $\vec{F}$ in (2.1) is linear in $x_1, x_2, \ldots, x_n$ the dynamical system is called a linear dynamical system.*

This means the system acquires the simpler form

$$\boxed{\dot{\vec{x}} = \vec{F}(\vec{x}) = A\vec{x}}, \tag{2.7}$$

with A being an $n \times n$-matrix with constant entries, i.e. A_{ij} =const for $1 \leq i, j \leq n$. Furthermore,

Definition: *A linear system is called simple, if A is non-singular, i.e.* $\det A \neq 0$ *and A has non-zero eigenvalues.*

Linear systems have some very useful properties:

Proposition: *The only fixed point of a simple linear system is the origin.*

Proof: We show this for $n = 2$: Taking for this purpose the matrix A to be in the most general form with arbitrary constants entries $a, b, c, d \in \mathbb{R}$, equation (2.7) for the fixed point becomes

$$\vec{F}(\vec{x}_f) = A\vec{x}_f = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0. \tag{2.8}$$

From this follows

$$\left.\begin{array}{l} ax_1 + bx_2 = 0 \Leftrightarrow x_1 = -b/ax_2 \\ cx_1 + dx_2 = 0 \Leftrightarrow x_1 = -d/cx_2 \end{array}\right\} \Rightarrow (\det A)\, x_2 = 0 \tag{2.9}$$

Since the determinant of A is non-vanishing we conclude from the last equality in (2.9) that $x_2 = 0$. A similar argument leads to $x_1 = 0$. As there are no further solutions to (2.8), the only fixed point of this linear system is the origin $\square$.

2.2.1 Change of variables, similarity classes

An important concept and technique is to change the variables $\vec{x}$ to some new set of variables $\vec{y}$. We achieve this via the transformation

$$\vec{x} = U\vec{y} \qquad \text{with } U_{ij} \in \mathbb{R} \qquad 1 \leq i, j \leq n. \tag{2.10}$$

Substituting this expression into the equation for the linear system (2.7), it the follows that

$$U\dot{\vec{y}} + \dot{U}\vec{y} = AU\vec{y}. \tag{2.11}$$

Assuming that the inverse U^{-1} of U exists, together with the fact that $\dot{U} = 0$, we obtain from (2.11) a new linear system

$$\dot{\vec{y}} = B\vec{y} =: \vec{Y}(\vec{y}) \qquad \text{with } \; B = U^{-1}AU. \tag{2.12}$$

Therefore we conclude that if two matrices A and B are in the same *similarity class (equivalence class)*, i.e. related by a similarity transformation $B = U^{-1}AU$, the solutions to their corresponding linear systems can be obtained from each other simply by relating the corresponding variables according to (2.10).

We recall now an important fact: The set of all $n \times n$-matrices can be decomposed into a finite set of similarity classes. We can use these classes to characterize the solutions of the corresponding linear systems as all members belonging to one particular class exhibit the same qualitative behaviour.

We present this here for the case $n = 2$. When starting with a matrix A we have to bring it first into the Jordan form by means of a similarity transformation $J = U^{-1}AU$. It is then sufficient to discuss the qualitative behaviour for all possible systems related to the J's, which are completely determined by their eigenvalues. For a generic 2×2-matrix we compute the eigenvalues from the characteristic equation

$$\det \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = \lambda^2 - \lambda ad + ad - bc = \lambda^2 - \lambda \operatorname{tr} A + \det A = 0, \tag{2.13}$$

which means the two eigenvalues are

$$\lambda_\pm = \frac{1}{2}\left(\operatorname{tr} A \pm \sqrt{\Delta}\right) \qquad \text{with } \Delta = (\operatorname{tr} A)^2 - 4 \det A. \tag{2.14}$$

There are precisely ten different cases (similarity classes), whose qualitative behaviour we list at first without any proof. The possibilities are

$\Delta > 0 \equiv$ **real eigenvalues** $J = \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix}$

i) $\lambda_+ > \lambda_- > 0 \equiv$ unstable node

ii) $\lambda_- < \lambda_+ < 0 \equiv$ stable node

iii) $\lambda_- < 0 < \lambda_+ \equiv$ saddle point

$\Delta = 0 \equiv$ **equal eigenvalues**

diagonal J :

iv) $\lambda_0 > 0 \equiv$ unstable star node

v) $\lambda_0 < 0 \equiv$ stable star node

non-diagonal J :

vi) $\lambda_0 > 0 \equiv$ unstable improper node

vii) $\lambda_0 < 0 \equiv$ stable improper node

$\Delta < 0 \equiv$ **complex eigenvalues** $J = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}, \alpha \in \mathbb{R}, \beta \in \mathbb{R}^+, \lambda_\pm = \alpha \pm i\beta$

viii) $\alpha > 0 \equiv$ unstable focus

ix) $\alpha = 0 \equiv$ centre

x) $\alpha < 0 \equiv$ stable focus

See figure 4 for the relevant phase portraits.

We can organize the ten cases into four basic groups: stable: $\{ii), v), vii), x)\}$, unstable: $\{i), iv), vi), viii)\}$, saddle: $\{iii)\}$ and centre:$\{ix)\}$.

Proof: Here we prove the qualitative behaviour only for the two cases *i)* and *ii)*. First we need to show that J is of diagonal form. From the eigenvalue equation

$$A\vec{v}_\pm = \lambda_\pm \vec{v}_\pm \qquad \text{with } \lambda_+ \neq \lambda_- \tag{2.15}$$

we construct a matrix U which consists of the eigenvectors of A as column vectors

$$U = (\vec{v}_+, \vec{v}_-). \tag{2.16}$$

Then we compute

$$AU = (A\vec{v}_+, A\vec{v}_-) = (\lambda_+\vec{v}_+, \lambda_-\vec{v}_-) = UJ. \tag{2.17}$$

Since $\lambda_+ \neq \lambda_-$ the two eigenvalues $\vec{v}_+$ and $\vec{v}_-$ are linearly independent, such that the matrix U is nonsingular, i.e. $\det U \neq 0$. Therefore the inverse U^{-1} exists, such that

$$J = U^{-1}AU = \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix}. \tag{2.18}$$

Next we consider the dynamical system produced by the matrix J

$$\begin{pmatrix} \dot{y}_+ \\ \dot{y}_- \end{pmatrix} = \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix} \begin{pmatrix} y_+ \\ y_- \end{pmatrix}, \tag{2.19}$$

which is evidently solved by

$$y_\pm = k_\pm e^{\lambda_\pm t}, \tag{2.20}$$

such that

$$y_- = k_-(y_+/k_+)^{\lambda_+/\lambda_-}. \tag{2.21}$$

Clearly if $\lambda_+ > \lambda_- > 0$ we obtain from this the phase portrait for case *i)* and when $\lambda_- < \lambda_+ < 0$ we obtain the phase portrait for case *ii)*. Similar arguments hold for the remaining cases *iii)*-*x)*, which we leave as exercises.

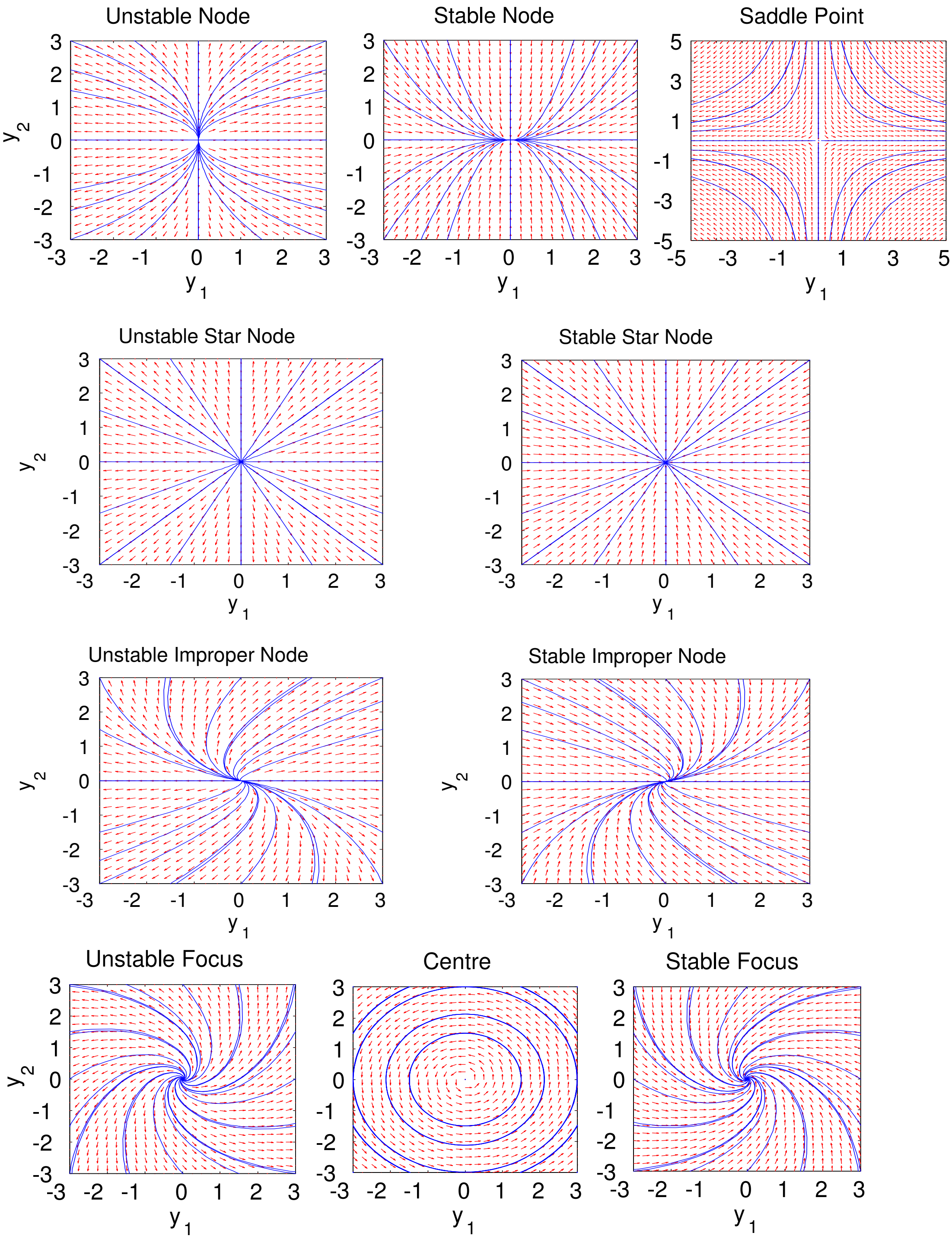

Figure 4: Phase portraits for the ten different equivalence classes.

We conclude this section by assembling some more useful terminology:

Definition: *A separatrix is a trajectory which divides the phase space into regions with distinctly different types of qualitative behaviour. It crosses a fixed point tangentially to a fixed radial direction. A stable (unstable) separatrix approaches (leaves) the fixed point for increasing time.*

Definition: *The directions of the straight line separatrix are called principal directions. They are the eigenvectors of the linear system A.*

Definition: *An isocline is a curve in phase space on which the trajectories have constant gradient, i.e.*

$$\frac{dx_2}{dx_1} = \frac{\dot{x}_2}{\dot{x}_1} = \frac{F_2(x_1, x_2)}{F_1(x_1, x_2)} = \kappa \in \mathbb{R}. \tag{2.22}$$

In particular we have $\kappa = 0$ for $F_2(x_1, x_2) = 0$ and $\kappa \to \infty$ for $F_1(x_1, x_2) = 0$. The separatrices, isoclines and principle directions can be used to include information into the phase portrait without knowing the explicit solution.

3. Analysis of nonlinear second order differential equations

Consider a linear second order differential equation of the general form

$$\ddot{x} + a\dot{x} + bx + c = 0, \qquad a, b, c \in \mathbb{R}. \tag{3.1}$$

Introducing the new variables

$$x_1 := x \qquad \text{and} \qquad x_2 := \dot{x}, \tag{3.2}$$

the equation (3.1) is converted into a set of coupled first order differential equations

$$\dot{x}_1 = x_2, \tag{3.3}$$

$$\dot{x}_2 = -ax_2 - bx_1 - c, \tag{3.4}$$

which is called the *standard form* of the second order differential equation (3.1). Notice that this is easily extended to nonlinear second order differential equations of the form

$$\ddot{x} + \sum_{n=0}^{p} \sum_{m=0}^{q} a_{nm} \dot{x}^n x^m = 0, \qquad a_{nm}, \in \mathbb{R}, p, q \in \mathbb{N}. \tag{3.5}$$

For $a_{00} = c$, $a_{10} = a$ and $a_{01} = b$ the equation (3.5) reduces to (3.1). Using the same parameterization as in (3.2) the nonlinear second order differential equation (3.5) converts into a set of two coupled first order nonlinear differential equations

$$\dot{x}_1 = x_2, \tag{3.6}$$

$$\dot{x}_2 = -\sum_{n=0}^{p} \sum_{m=0}^{q} a_{nm} x_2^n x_1^m. \tag{3.7}$$

This is one of the main motivations why we are interested in $n = 2$ of the system (2.1).

3.1 Linearization at a fixed point

Let us now focus on a dynamical system with $n = 2$, that is

$$\dot{x}_i = F_i(x_1, x_2) \qquad \text{for } 1 \leq i \leq 2. \tag{3.8}$$

with the additional assumption that the functions $F_i(x_1, x_2)$ are continuously differentiable in some neighbourhood of the point (x_1^0, x_2^0). We can now carry out a Taylor expansion about the point (x_1^0, x_2^0)

$$F_i(x_1, x_2) = F_i(x_1^0, x_2^0) + (x_1 - x_1^0)\frac{\partial F_i(x_1^0, x_2^0)}{\partial x_1} + (x_2 - x_2^0)\frac{\partial F_i(x_1^0, x_2^0)}{\partial x_2} + R_i(x_1, x_2). \tag{3.9}$$

In order to ensure convergence of the series we make the following assumption on the remainder function $R_i(x_1, x_2)$

$$\lim_{r \to 0} \frac{R_i(x_1, x_2)}{r} = 0 \qquad \text{for } r = \sqrt{(x_1 - x_1^0)^2 + (x_2 - x_2^0)}. \tag{3.10}$$

Taking now (x_1^0, x_2^0) to be a fixed point for the system (3.8), we can re-write (3.9) as

$$\dot{x}_i = (x_1 - x_1^0)\frac{\partial F_i(x_1^0, x_2^0)}{\partial x_1} + (x_2 - x_2^0)\frac{\partial F_i(x_1^0, x_2^0)}{\partial x_2} + R_i(x_1, x_2). \tag{3.11}$$

Next we introduce the new variable $y_i = x_i - x_i^0$, such that (3.11) becomes a Maclaurin expansion

$$\dot{y}_i = y_1\frac{\partial F_i(x_1^0, x_2^0)}{\partial x_1} + y_2\frac{\partial F_i(x_1^0, x_2^0)}{\partial x_2} + R_i(y_1 + x_1^0, y_2 + x_2^0). \tag{3.12}$$

Neglecting now R_i, the system (3.12) constitutes the linearization of the system (3.8) at the fixed point $\vec{x}_f = (x_1^0, x_2^0)$, i.e. we can write the system as

$$\boxed{\dot{\vec{y}} = A\vec{y} \qquad \text{with } A = \left.\begin{pmatrix} \frac{\partial X_1}{\partial x_1} & \frac{\partial X_1}{\partial x_2} \\ \frac{\partial X_2}{\partial x_1} & \frac{\partial X_2}{\partial x_2} \end{pmatrix}\right|_{\vec{x}_f = (x_1^0, x_2^0)}}. \tag{3.13}$$

Here A is the *Jacobian matrix* of the vectorfield $\vec{F}$ in $\vec{x}_f = (x_1^0, x_2^0)$.

3.1.1 Examples for linearizations

In order to exemplify the working of the above concept let us consider two examples:

i) Find the linearization for the dynamical system

$$\dot{x}_1 = x_2^2 + \sin x_1 - a^2 \qquad a \in \mathbb{R} \tag{3.14}$$

$$\dot{x}_2 = \sinh(x_2 - a). \tag{3.15}$$

Solution: The fixed points are found by setting the right hand sides of (3.14) and (3.15) to zero, which means we have to solve

$$\sinh(x_2 - a) = 0 \quad \Rightarrow \quad x_2 = a, \tag{3.16}$$

substituting this into (3.14) gives

$$\sin x_1 = 0 \quad \Rightarrow \quad x_1 = n\pi. \tag{3.17}$$

Therefore we find an infinite amount of fixed points $\vec{x}_f^{(n)} = (n\pi, a)$.

To obtain the linearization, we compute first the Jacobian matrix from (3.13)

$$A(x_1, x_2) = \begin{pmatrix} \cos x_1 & 2x_2 \\ 0 & \cosh(x_2 - a) \end{pmatrix} \Rightarrow A(\vec{x}_f^{(n)}) = \begin{pmatrix} (-1)^n & 2a \\ 0 & 1 \end{pmatrix}. \tag{3.18}$$

Recalling that $\dot{\vec{y}} = A\vec{y}$, we obtain the following linearization for the system (3.14), (3.15)

$$\dot{y}_1^{(n)} = (-1)^n y_1^{(n)} + 2a y_2^{(n)} \tag{3.19}$$
$$\dot{y}_2^{(n)} = y_2^{(n)}, \tag{3.20}$$

where $y_1^{(n)} = x_1 - n\pi$, $y_2^{(n)} = x_2 - a$.

ii) At the origin find the linearization for the dynamical system

$$\dot{x}_1 = x_1^5 \exp(2x_2) \tag{3.21}$$
$$\dot{x}_2 = x_2[\exp(4x_1) - 1] \tag{3.22}$$

Solution: Clearly the origin is a fixed point $\vec{x}_f = (0,0)$.

To obtain the linearization, we compute the Jacobian matrix

$$A(x_1, x_2) = \begin{pmatrix} 5x_1^4 e^{2x_2} & 2x_1^5 e^{2x_2} \\ 4x_2 e^{4x_1} & e^{4x_1} - 1 \end{pmatrix} \Rightarrow A(\vec{x}_f) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.23}$$

We observe that this yields a non-simple linearization, i.e. $\det A = 0$, which demonstrates some of the limitations of the linearization procedure. The linearized system at the fixed point is now simply $\dot{x}_1 = \dot{x}_2 = 0$.

To make these observations more precise we discuss now the linearization theorem.

3.2 The linearization theorem

In the previous section we have seen how a nonlinear system may be approximated in form of a linear system near a fixed point. The following theorem will make it more precise what type of information we can extract from the linear system for the nonlinear one.

Linearization theorem: *Consider a nonlinear system which possesses a simple linearization at some fixed point. Then in a neighbourhood of the fixed point the phase portraits of the linear system and its linearization are qualitatively equivalent, if the eigenvalues of the Jacobian matrix have a nonzero real part, i.e. the linearized system is not a centre.*

Proof: The proof is omitted here[1].

[1] See e.g. P. Hartman, Ordinary differential equations, (Wiley, New York 1964)

Comments: The linearization theorem constitutes the basis for stability analysis. The theorem does not apply for nonsimple systems, i.e. $\det A = 0$, and those for which the eigenvalues $\lambda_{1/2}$ of A are purely imaginary, i.e. $\lambda_{1/2} \in i\mathbb{R}$.
The notion of qualitative equivalence includes:

- the classification of the fixed point as stable or unstable
- the classification of the fixed points as nodes, foci or saddle points
- the direction in which the trajectories run in or out of the fixed point

3.2.1 Examples for linearization theorem

We discuss now three examples i) a "good" one for which the linearization theorem can be applied, and two related to the exemptions of the theorem ii) one for which the eigenvalues of J are purely imaginary, which therefore can not be analyzed by means of the linearization theorem and iii) one which possesses a nonsimple linearization, such that also in this case no information can be extracted from the linearization theorem.

i) Find the linearization for the dynamical system

$$\dot{x}_1 = x_1 + 4x_2 + \exp(x_1) - 1 \tag{3.24}$$

$$\dot{x}_2 = -x_2 - x_2 \exp(x_1) \tag{3.25}$$

at the origin.

Solution: Clearly the origin is a fixed point $\vec{x}_f = (0, 0)$.

To obtain the linearization, we compute the Jacobian matrix

$$A(x_1, x_2) = \begin{pmatrix} 1 + e^{x_1} & 4 \\ -x_2 e^{x_1} & -(e^{x_1} + 1) \end{pmatrix} \Rightarrow A(\vec{x}_f) = \begin{pmatrix} 2 & 4 \\ 0 & -2 \end{pmatrix}. \tag{3.26}$$

Let us first compute the eigenvalues for A from

$$\det(A - \lambda\mathbb{I}) = \begin{vmatrix} 2-\lambda & 4 \\ 0 & -2-\lambda \end{vmatrix} = \lambda^2 - 4 = 0 \quad \Rightarrow \lambda_\pm = \pm 2. \tag{3.27}$$

Since we have two real eigenvalues of opposite sign, we expect a saddle point for the linearized system at the origin. However, the matrix A is not quite in the Jordan normal form and we have to perform a similarity transformation to achieve this. Only in this way we can make a valid statement about the precise qualitative picture of the original system (3.24) and (3.25). For this we seek a matrix U which solves the equation

$$U^{-1} A U = J = \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix}. \tag{3.28}$$

As discussed above we can take $U = (\vec{v}_+, \vec{v}_-)$ for this purpose, where $\vec{v}_+, \vec{v}_-$ are the eigenvectors of A. We determine the eigenvectors[2] from

$$A\vec{v} = \pm 2\vec{v} \quad \Rightarrow \vec{v}_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \vec{v}_- = \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{3.29}$$

and verify that indeed[3]

$$\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}. \tag{3.30}$$

Recalling now that

$$\dot{\vec{x}} = A\vec{x}, \qquad \dot{\vec{y}} = U^{-1}AU\vec{y} = J\vec{y}, \qquad \vec{x} = U\vec{y}, \tag{3.31}$$

we may find the phase portrait for the linearized system (3.24), (3.25) by transforming the standard phase portrait related to the Jordan normal form (3.28). We may do this by transforming some characteristic points, which we label by $X_0, \ldots, X_5$ as depicted in the phase portraits of figure 4

$$X_0 : \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ -a' \end{pmatrix} = \begin{pmatrix} a' \\ -a' \end{pmatrix}, \qquad X_1 : \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -a' \\ a' \end{pmatrix} = \begin{pmatrix} 0 \\ -a' \end{pmatrix},$$
$$X_2 : \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -a' \\ -a' \end{pmatrix} = \begin{pmatrix} -2a' \\ a' \end{pmatrix}, \qquad X_3 : \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a' \\ -a' \end{pmatrix} = \begin{pmatrix} 0 \\ a' \end{pmatrix},$$
$$X_4 : \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a' \\ a' \end{pmatrix} = \begin{pmatrix} 2a' \\ -a' \end{pmatrix}, \qquad X_5 : \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a' \\ 0 \end{pmatrix} = \begin{pmatrix} a' \\ 0 \end{pmatrix},$$

where we parameterize the points by $a' \in \mathbb{R}$. In this way we construct the phase portraits in the (x_1, x_2)-plane from the one in the (y_1, y_2)-plane as depicted in figure 4.

We observe that $\vec{v}_+$ and $\vec{v}_-$, i.e. the eigenvectors of A, constitute an unstable separatrix and a stable separatrix, respectively. Next we compare the outcome of the linearized system with the phase portrait of the nonlinear system which we depict in figure 5. We observe that only in the neighbourhood of the origin the phase portraits of the nonlinear system and its linearization are qualitatively equivalent. To see this we look at the original nonlinear system (3.24), (3.25).

[2] Recall also that eigenvectors are only fixed up to an overall factor. This is easily seen as follows: Multiply the eigenvalue equation $A\vec{v} = \lambda\vec{v}$ by some constant κ

$$\kappa A\vec{v} = \kappa\lambda\vec{v} \Rightarrow A\kappa\vec{v} = \lambda\kappa\vec{v} \Leftrightarrow A\vec{w} = \lambda\vec{w},$$

which means that the vector $\vec{w} = \kappa\vec{v}$ is also an eigenvector of A to the same eigenvalue λ.

[3] Recall that for a general 2×2-matrix $U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ the inverse is computed to $U = \frac{1}{\det U} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

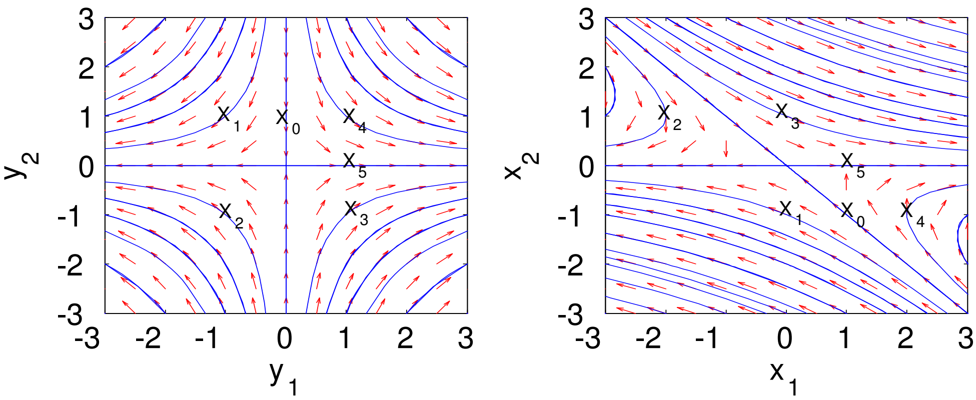

Figure 5: Phase portraits for the linear systems $\dot{\vec{y}} = J\vec{y}$ and $\dot{\vec{x}} = A\vec{x}$.

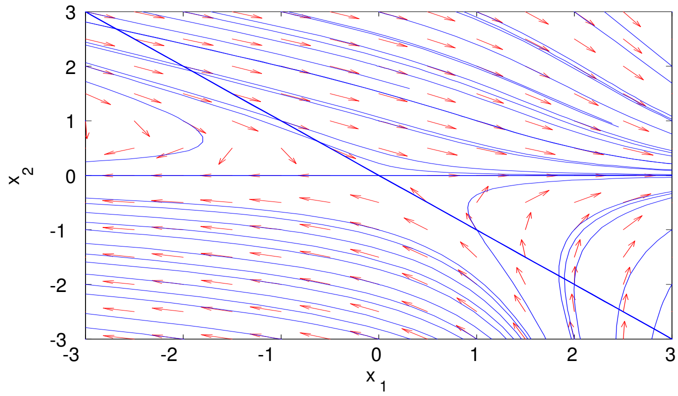

Figure 6: Phase portraits for the nonlinear system (3.24)-(3.25) .

For $x_2 = 0$ it follows $\dot{x}_2 = 0$, which means that the unstable separatrix remains the same in the nonlinear system. Next we investigate how the nonlinear system behaves near the stable separatrix $x_2 = -x_1$. For this we compute the gradient a tiny bit away from the line

$$
\begin{aligned}
\frac{dx_2}{dx_1} &= \frac{\dot{x}_2}{\dot{x}_1} = \frac{-x_2 - x_2\exp(x_1)}{x_1 + 4x_2 + \exp(x_1) - 1} && (3.32)\\
&= \frac{x_1(1+e^{x_1})}{-3x_1 + e^{x_1} - 1} \qquad \text{for } x_2 = -x_1 && (3.33)\\
&\approx \frac{x_1(2+x_1)}{-2x_1} \qquad \text{with } e^{x_1} \approx (1+x_1) && (3.34)\\
&= -1 - x_1/2 && (3.35)\\
&= \begin{cases} > -1 & \text{for } \ x_1 < 0 \\ < -1 & \text{for } \ x_1 > 0 \end{cases} && (3.36)
\end{aligned}
$$

This means near the origin the trajectory of the straight line is slightly bended down.

ii) Find the linearization for the two dynamical systems (one for the + sign and one for the − sign)

$$\dot{x}_1 = -x_2 \pm x_1(x_1^2 + x_2^2) \tag{3.37}$$
$$\dot{x}_2 = x_1 \pm x_2(x_1^2 + x_2^2) \tag{3.38}$$

at the origin.

Solutions: Once again it is easy to see that the origin is indeed a fixed point $\vec{x}_f = (0, 0)$.

To obtain the linearization, we compute the Jacobian matrix

$$A(x_1, x_2) = \begin{pmatrix} \pm 3x_1^2 \pm x_2^2 & -1 \pm 2x_1x_2 \\ 1 \pm 2x_1x_2 & \pm x_1^2 \pm 3x_2^2 \end{pmatrix} \Rightarrow A(\vec{x}_f) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{3.39}$$

The eigenvalues for A are computed as

$$\det(A - \lambda\mathbb{I}) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0 \quad \Rightarrow \lambda_\pm = \pm i. \tag{3.40}$$

This means the eigenvalues of the Jacobian matrix at $\vec{x}_f$ have no real part and the linearization theorem does not apply. The linearized system is of the type ix) in our classification scheme, that is we have a centre near the origin. We further observe that the linearization is the same for both systems independent of the chosen sign. We recall that the phase portrait of a centre exhibits rotational symmetry, which makes it suggestive to introduce polar coordinates

$$x_1 = r\cos\vartheta \qquad \text{and} \qquad x_2 = r\sin\vartheta. \tag{3.41}$$

Differentiating this and on the other hand substitution into (3.37), (3.38) yields

$$\dot{x}_1 = \dot{r}\cos\vartheta - r\sin\vartheta\dot{\vartheta} = -r\sin\vartheta \pm r^3\cos\vartheta \tag{3.42}$$
$$\dot{x}_2 = \dot{r}\sin\vartheta + r\cos\vartheta\dot{\vartheta} = r\cos\vartheta \pm r^3\sin\vartheta. \tag{3.43}$$

We easily read off that

$$\dot{r} = \pm r^3 \qquad \text{and} \qquad \dot{\vartheta} = 1 \tag{3.44}$$

solve (3.42) and (3.43). Note further that the different signs in (3.44) are not visible in the linear system, i.e. $r^3 \to 0$, as we already saw. We conclude now from (3.44)

$$+\text{sign:} \quad \dot{r} > 0 \;\; \forall\, r > 0, \; \dot{\vartheta} = \text{const} \quad \Rightarrow \text{ trajectories spiral outwards as } t \to \infty$$
$$-\text{sign:} \quad \dot{r} < 0 \;\; \forall\, r > 0, \; \dot{\vartheta} = \text{const} \quad \Rightarrow \text{ trajectories spiral inwards as } t \to \infty$$

As $\dot{\vartheta}$ is positive the trajectories spiral in the positive mathematical sense (≡ anti-clockwise). Therefore we deduce the qualitative behaviour as depicted in figure 6.

Note that we have been able to make these deductions simply from (3.44) and have not applied the linearization theorem.

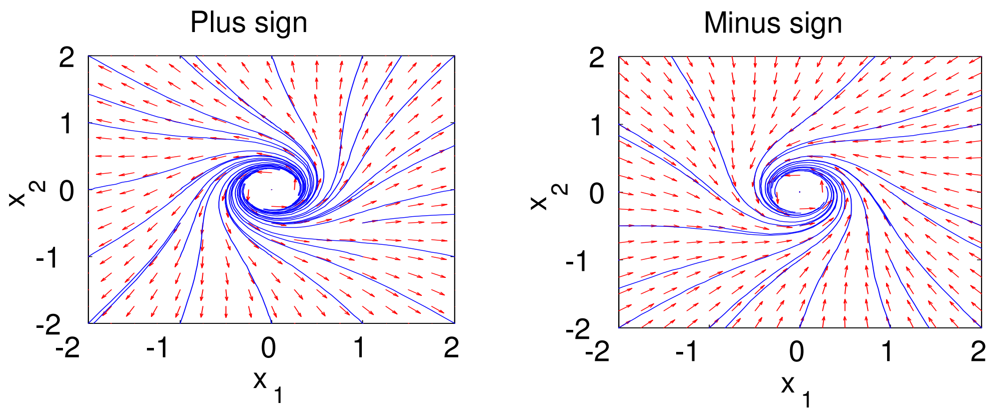

Figure 7: Phase portraits for the nonlinear systems (3.37), (3.38).

iii) Find the linearization for the dynamical system

$$\dot{x}_1 = x_1^2 + 2x_1x_2 \tag{3.45}$$

$$\dot{x}_2 = x_2^2 + 2x_1x_2 \tag{3.46}$$

at the origin.

Solution: Clearly the origin is once again a fixed point $\vec{x}_f = (0,0)$.

To obtain the linearization, we compute the Jacobian matrix

$$A(x_1,x_2) = \begin{pmatrix} 2x_1 + 2x_2 & 2x_1 \\ 2x_1 & 2x_1 + 2x_2 \end{pmatrix} \Rightarrow A(\vec{x}_f) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.47}$$

Therefore $\det A = 0$, which means this is a nonsimple system and the linearization theorem does not apply. Despite this we can make some observations and still obtain a phase portrait: Clearly the linear system is $x_i(t) = c_i$ for $i = 1,2$ and $c_i \in \mathbb{R}$. The system is symmetric around $x_1 = x_2$, which reduces the analysis by a factor $1/2$. The isoclines are computed as

$$\frac{dx_2}{dx_1} = \frac{x_2^2 + 2x_1x_2}{x_1^2 + 2x_1x_2} = \begin{cases} 0 & \text{for } \; x_2 = 0 \\ \infty & \text{for } \; x_1 = 0 \\ 1 & \text{for } \; x_1 = x_2. \end{cases} \tag{3.48}$$

Let us next see what happens near the line $(x_1, 0)$. Assuming for this that x_2/x_1 is small we can write

$$\begin{aligned} \frac{dx_2}{dx_1} &= \frac{(x_2/x_1)^2 + 2x_2/x_1}{1 + 2x_2/x_1} = \left[\frac{x_2^2}{x_1^2} + 2\frac{x_2}{x_1}\right]\left[1 - 2\frac{x_2}{x_1} + 4\left(\frac{x_2}{x_1}\right)^2 + \mathcal{O}\left[\left(\frac{x_2}{x_1}\right)^3\right]\right] \\ &= 2\frac{x_2}{x_1} - 3\left(\frac{x_2}{x_1}\right)^2 + \mathcal{O}\left[\left(\frac{x_2}{x_1}\right)^3\right]. \end{aligned}$$

Near the line $(0, x_2)$ we assume for this that x_1/x_2 to be small such that we may write

$$\begin{aligned}\frac{dx_2}{dx_1} = \frac{1 + x_2/2x_1}{1 + x_1/2x_2} &= \left[1 + \frac{x_2}{2x_1}\right]\left[1 - \frac{x_1}{2x_2} + \left(\frac{x_1}{2x_2}\right)^2 + \mathcal{O}\left[\left(\frac{x_1}{2x_2}\right)^3\right]\right] \\ &= \frac{x_2}{2x_1} + \frac{3}{4} - \frac{x_1}{2x_2} + \mathcal{O}\left[\left(\frac{x_1}{2x_2}\right)^3\right].\end{aligned}$$

At the lines $(-x_1, x_2)$ and $(x_1, -x_2)$ we find

$$\frac{dx_2}{dx_1} = -1. \tag{3.49}$$

Next we determine the direction of time. Recall for this the simple example $\dot{x}_1 = -x_1$, $\dot{x}_2 = -x_2$ which can be solved explicitly by $x_i(t) = k_i \exp(-t)$ with $k_i \in \mathbb{R}$ and $i = 1, 2$. Therefore in $x_2 = k_2/k_1 x_1$ the information concerning the direction of time is lost. We can either study the limit $\lim_{t\to\infty} x_i(t)$ or alternatively determine the regions in which $\dot{x}_i(t) < 0$ or $\dot{x}_i(t) > 0$. Let us do this for the system (3.45), (3.46). We determine the regions

$$\begin{aligned} \dot{x}_1 > 0 \quad &\Leftrightarrow \quad x_1(x_1 + 2x_2) > 0 \quad \begin{array}{l} \Leftrightarrow \quad x_1 > 0 \quad \wedge \quad x_2 > -x_1/2 \\ \Leftrightarrow \quad x_1 < 0 \quad \wedge \quad x_2 < -x_1/2 \end{array} \\ \dot{x}_2 > 0 \quad &\Leftrightarrow \quad x_2(x_2 + 2x_1) > 0 \quad \begin{array}{l} \Leftrightarrow \quad x_2 > 0 \quad \wedge \quad x_2 > -2x_1 \\ \Leftrightarrow \quad x_2 < 0 \quad \wedge \quad x_2 < -2x \end{array} \end{aligned} \tag{3.50}$$

Assembling all this information we obtain the phase portrait of figure 8.

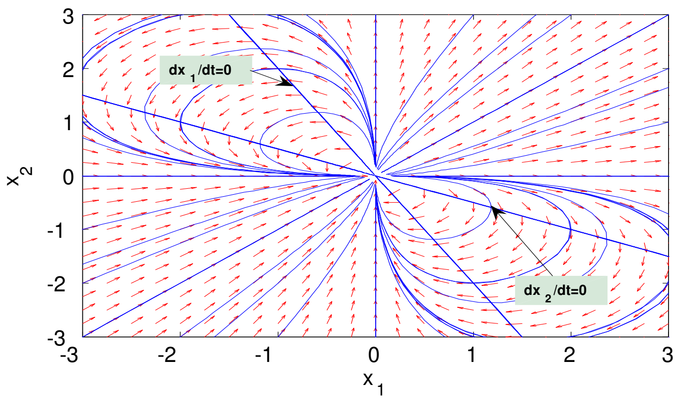

Figure 8: Phase portraits for the nonlinear systems (3.45), (3.46).

3.3 Stability of fixed points

Let us now slightly enlarge the notion of stability of fixed points and also include into it some information about the neighbourhood around it.

Definition: *A fixed point $\vec{x}_f$ of the system $\dot{\vec{x}} = \vec{F}(\vec{x})$ is said to be* <u>stable</u> *(in the sense of Lyapunov), if for every neighbourhood $N(\vec{x}_f)$ there exists a smaller neighbourhood $N'(\vec{x}_f) \subseteq N(\vec{x}_f)$, such that every trajectory which passes though $N'(\vec{x}_f)$ remains in $N(\vec{x}_f)$ as the time t increases.*

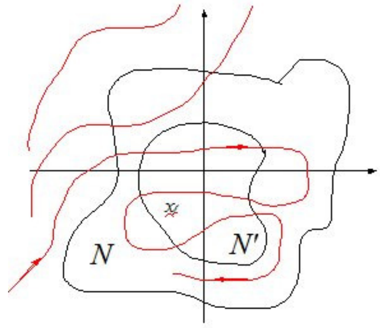

Figure 9: Stability in the sense of Lyapunov

Definition: *A fixed point $\vec{x}_f$ of the system $\dot{\vec{x}} = \vec{F}(\vec{x})$ is said to be* <u>asymptotically stable</u>*, if there exists a neighbourhood $N(\vec{x}_f)$ of $\vec{x}_f$ such that every trajectory passing through $N(\vec{x}_f)$ approaches $\vec{x}_f$ as $t \to \infty$.*

Definition: *A* <u>neutrally stable</u> *fixed point $\vec{x}_f$ is a fixed point which is stable but not asymptotically stable.*

Definition: *An* <u>unstable</u> *fixed point $\vec{x}_f$ is a fixed point which is not stable.*

Some examples for these definitions are the following fixed points:

a) Stable nodes are asymptotically stable.

b) Unstable nodes are unstable fixed points.

c) Centres are neutrally stable.

We can deduce a) and b) from the linearization theorem, but not c)!

In the previous examples ii) and iii) we saw already that the linearization theorem is only of limited use. As alternative method which can also be applied for these cases one seeks so-called *Lyapunov function*. This method will also cover the treatment of nonsimple fixed points and those which are centres. In addition, these functions will provide also provide information which goes beyond the treatment of the isolated fixed point and yields in also a *domain of stability*.

3.4 Lyapunov functions (stability theorem)

Before being very precise, we get first some intuitive understanding by looking at an example: We saw already that the origin is a centre for the system in example ii), that is (3.37), (3.38) and the linearization theorem can not be applied in that case. Now we consider a function

$$V[\vec{x}(t)] = V[x_1(t), x_2(t)] = x_1^2(t) + x_2^2(t), \tag{3.51}$$

depending on the solution $\vec{x}(t)$ of the system (3.37), (3.38) and address the following question: How does V change along the trajectories of this dynamical system? To see this we compute

$$\begin{aligned}
\dot{V} &= \frac{dV}{dt} = \frac{\partial V}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial V}{\partial x_2}\frac{dx_2}{dt} = \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2, \\
&= 2x_1\left[-x_2 \pm x_1(x_1^2 + x_2^2)\right] + 2x_2\left[x_1 \pm x_2(x_1^2 + x_2^2)\right], \\
&= \pm 2(x_1^2 + x_2^2)^2 = \begin{cases} > 0 \\ < 0 \end{cases} \quad \text{for } \vec{x} \neq (0,0).
\end{aligned}$$

Therefore we conclude for the plus sign that $V[\vec{x}(t)]$ is an increasing function on any trajectory and $\lim_{t\to\infty} V \to \infty$. Since $V[\vec{x}(t)] = |\vec{x}(t)|^2$ this means that $\vec{x}(t) \to \infty$ for $t \to \infty$ such that the origin is an unstable fixed point. On the other hand, for the minus sign we conclude that $V[\vec{x}(t)]$ is a decreasing function on any trajectory and $\lim_{t\to\infty} V \to 0$. Therefore in this case we have $\vec{x}(t) \to 0$ for $t \to \infty$, which implies that now the origin is an asymptotically stable fixed point. This behaviour is precisely what we have concluded already before when we investigated this example. Let us make these observations more rigorous. For this we need to define first a few notions.

Definition: *A real valued function $f(\vec{x})$ is said to be* <u>positive (negative) definite</u> *in a neighbourhood $N(\vec{x}=0)$ if*

$$i)\;\; f(\vec{x}) = 0 \quad \text{and} \quad ii)\;\; f(\vec{x})>0 \;(\; f(\vec{x})<0) \qquad \text{for } \vec{x} \in N(\vec{x}=0)\backslash \vec{x}=0.$$

Definition: *A real valued function $f(\vec{x})$ is said to be* <u>positive (negative) semi-definite</u> *in a neighbourhood $N(\vec{x}=0)$ if*

$$i)\;\; f(\vec{x}) = 0 \quad \text{and} \quad ii)\;\; f(\vec{x})\geq 0 \;(f(\vec{x})\leq 0) \qquad \text{for } \vec{x} \in N(\vec{x}=0)\backslash\{\vec{x}=0\}.$$

We are now in the position to formulate the main theorem of this section:

Lyapunov stability theorem: *Consider the system $\dot{\vec{x}} = \vec{F}(\vec{x})$ with a fixed point at the origin. If there exists a real valued function $V(\vec{x})$ in a neighbourhood $N(\vec{x}=0)$ such that:*

i) *the partial derivatives $\partial V/\partial x_1$, $\partial V/\partial x_2$ exist and are continuous,*

ii) *the function $V(\vec{x})$ is positive definite,*

iii) *dV/dt is negative semi-definite (definite),*

then the origin is a stable (asymptotically stable) fixed point.

Proof: First we prove stability: From i) and ii) follows that the level curves of V form a continuum of closed curves around the origin.

$\therefore$ $\exists$ a real number $\kappa > 0$ such that $N' = \{\vec{x}|V(\vec{x}) < \kappa\} \subseteq N(\vec{x} = \vec{0})$

$\therefore$ it follows by iii) that if $\vec{x}' \in N'\backslash\{0\}$ then $\dot{V}(\varphi_t(\vec{x}')) \leq 0 \;\forall\; t \geq 0$[4]

$\therefore$ $\dot{V}(\varphi_t(\vec{x}')) \leq 0$ is a non-increasing function of t

$\therefore$ $V(\varphi_t(\vec{x}')) \leq \kappa \;\forall\; t \geq 0$

$\therefore$ $\varphi_t(\vec{x}') \in N' \;\forall\; t \geq 0$

$\therefore$ the origin is a stable fixed point $\square$.

Next we prove asymptotical stability when $\dot{V}$ is negative definite.

Now $V(\varphi_t(\vec{x}'))$ is a strictly decreasing function of t.

[4]Here φ is a *flow* on $\mathbb{R}$, which is a continuous map $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$, with $\varphi_0(\vec{x}) = \vec{x}$ for all $\vec{x} \in \mathbb{R}^2$ and $\varphi_{t_2} \circ \varphi_{t_1}(\vec{x}) = \varphi_{t_1+t_2}(\vec{x})$.

$\therefore V(\varphi_{t_2}(\vec{x}')) - V(\varphi_{t_1}(\vec{x}')) < \kappa \ \forall \ t_2 > t_1 \geq 0$

$\therefore$ by the mean value theorem[5] it follows that there exist a sequence $\{\tau_i\}_{i=1}^{\infty}$ tending to infinity, such that $\dot{V}(\varphi_t(\vec{x}')) \to 0$ as $\tau_i \to \infty$

$\therefore$ this implies that $\varphi_{\tau_i}(\vec{x}') \to 0$ as $\tau_i \to \infty$, since $\dot{V}$ is negative definite

$\because V(\varphi_t(\vec{x}'))$ is a strictly decreasing we have that $V(\varphi_t(\vec{x}')) < V(\varphi_{\tau_i}(\vec{x}')) \ \forall \ t \geq \tau_i$

$\therefore$ the flows $\{\varphi_t(\vec{x}') | t > \tau_i\}$ lie inside the level curves of V containing $\varphi_{\tau_i}(\vec{x}')$

$\because$ this holds for all τ_i it follows that $\varphi_{\tau_i}(\vec{x}') \to 0$ as $\tau_i \to \infty$, such that $\varphi_t(\vec{x}') \to 0$ as $t \to \infty$

$\therefore$ the origin is an asymtptotically stable fixed point, as this holds for all $\vec{x}' \in N'$ $\square$.

Definition: *A function V for which the conditions i)-iii) hold with iii) (definite) semi-definite is called a (strong) weak Lyapunov function.*
Definition: *A domain of stability of an asymptotically stable point is a neighbourhood of the fixed point in which all trajectories approach the fixed point.*

3.4.1 Examples for the usage of the Lyapunov stability theorem:

i) Let us consider once more the system (3.37), (3.38) with the minus sign. As a candidate for a Lyapunov function we take again $V[\vec{x}(t)] = x_1^2 + x_2^2$. We have to check whether the conditions *i)*-*iii)* of the Lyapunov stability theorem hold for this function. Clearly the partial derivatives of V exist and are continuous. Since $V(0) = 0$ and $V(\vec{x}) > 0$ for $\vec{x} \neq 0$ the function $V[\vec{x}(t)]$ is a positive definite function for all $\vec{x} \in \mathbb{R}^2$. We already computed

$$\dot{V} = -2(x_1^2 + x_2^2) \tag{3.52}$$

Since $\dot{V}(0) = 0$ and $\dot{V}(\vec{x}) < 0$ for $\vec{x} \neq 0$ the function $\dot{V}[\vec{x}(t)]$ is a negative definite function for all $\vec{x} \in \mathbb{R}^2$. Therefore $V[\vec{x}(t)]$ is a strong Lyapunov function for all $\vec{x} \in \mathbb{R}^2$. We can apply the Lyapunov stability theorem and conclude that the origin is an asymptotically stable fixed point. Since $V(\vec{x}(t))$ is a strong Lyapunov function in the entire plane, the domain of stability is the entire plane. This behaviour is also confirmed in figure 6, where we observe that every trajectory ends up in the fixed point at the origin.

ii) Consider the dynamical system

$$\dot{x}_1 = -x_1 + x_1 x_2 \tag{3.53}$$

$$\dot{x}_2 = -x_2 + x_1 x_2. \tag{3.54}$$

Show that $V[\vec{x}(t)] = x_1^2 + x_2^2$ is a strong Lyapunov function for the origin of the above system. Find the domain of stability.

[5] Recall the mean value theorem: A function $f(x)$ is continuous and differentiable on some interval $[a, b]$, then there exits a ζ with $a < \zeta < b$ such that

$$\frac{f(b) - f(a)}{b - a} = f(\zeta).$$

Solution: Let us first see what we would expect from the linearization theorem. It is easy to see that the system has two fixed points $\vec{x}_f^{(1)} = (0,0)$ and $\vec{x}_f^{(2)} = (1,1)$. The Jacobian is computed to

$$A(x_1, x_2) = \begin{pmatrix} x_2 - 1 & x_1 \\ x_2 & x_1 - 1 \end{pmatrix} \Rightarrow A(\vec{x}_f^{(1)}) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, A(\vec{x}_f^{(2)}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (3.55)$$

From this we obtain that $A(\vec{x}_f^{(1)})$ has degenerate positive eigenvalues $\lambda_+ = \lambda_- = -1$, such that we have a stable star node at the origin. At $A(\vec{x}_f^{(2)})$ we have two real eigenvalues of opposite sign $\lambda_\pm = \pm 1$, such that we have a saddle point at the second fixed point $\vec{x}_f^{(2)} = (1,1)$.

Let us next see what we would conclude from the Lyapunov stability theorem. We saw already that the condition $i)$ and $ii)$ of the theorem hold for the function $V(\vec{x}(t))$. In order to check the third condition $iii)$ we compute next

$$\begin{aligned} \dot{V} &= \frac{dV}{dt} = \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2 \\ &= 2x_1(-x_1 + x_1 x_2) + 2x_2(-x_2 + x_1 x_2) \\ &= -2x_1^2(1 - x_2) - 2x_2^2(1 - x_1) \end{aligned}$$

$\therefore \dot{V}(\vec{x} = \vec{0}) = 0$ and $\dot{V}(\vec{x}) < 0$ for $x_1 < 1$ and $x_2 < 1$.

$\therefore \exists$ a neighbourhood of the origin in which $\dot{V}$ is negative definite.

$\therefore V[\vec{x}(t)]$ is a strong Lyapunov function.

$\therefore$ the origin is an asymptotically stable fixed point by the Lyapunov stability theorem.

In order to find the domain of stability $N' = \{\vec{x} | V(\vec{x}) < \kappa\}$ we need to determine a real number κ and the region in which $V[\vec{x}(t)]$ is a strong Lyapunov function. We saw that for $x_1 < 1$ and $x_2 < 1$ the function $V[\vec{x}(t)]$ is a strong Lyapunov function, which means we need to find κ from the relations

$$V[\vec{x}(t)] = x_1^2 + x_2^2 < \kappa \quad \wedge \quad x_1 < 1 \quad \wedge \quad x_2 < 1. \quad (3.56)$$

The smallest κ which satisfies these constraints is $\kappa = 1$. This means the domain of stability is $x_1^2 + x_2^2 < 1$. The conclusion we drew are confirmed by the phase portrait obtained from a numerical solution as depicted in figure 10.

$iii)$ Consider the dynamical system

$$\dot{x}_1 = x_1 x_2 - x_1^3 - x_2^2 \quad (3.57)$$
$$\dot{x}_2 = x_1 x_2 - x_1^2. \quad (3.58)$$

Show that $V(\vec{x}(t)) = x_1^2 + x_2^2$ is a weak Lyapunov function for the origin for the above system.

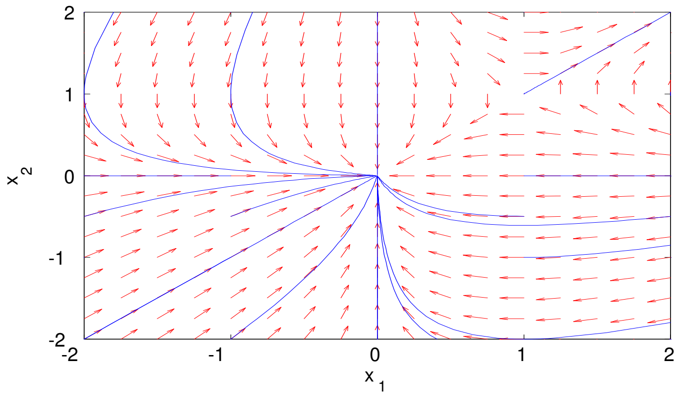

Figure 10: Phase portraits for the nonlinear systems (3.53), (3.54).

Solution: Obviously $\vec{x}_f = (0,0)$ is a fixed point for the system (3.57), (3.58). The linearization theorem does not apply in this case since the system is nonsimple. This is seen from the Jacobian matrix at the fixed point

$$A(\vec{x}_f) = \left.\begin{pmatrix} x_2 - 3x_1^2 & x_1 - 2x_2 \\ x_2 - 2x_1 & x_1 \end{pmatrix}\right|_{\vec{x}_f} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.59}$$

We found already that the condition *i*) and *ii*) of the theorem hold for the function $V[\vec{x}(t)]$. (Note that the conditions *i*) and *ii*) of the theorem are only concerned with the function V, but do not involve the actual dynamical system.) In order to test the condition *iii*) we compute next

$$\begin{aligned} \dot{V} &= \frac{dV}{dt} = \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2 \\ &= 2x_1(x_1x_2 - x_1^3 - x_2^2) + 2x_2(x_1x_2 - x_1^2) \\ &= -2x_1^4 \end{aligned}$$

$\therefore \dot{V}(\vec{x}=0) = 0$ and $\dot{V}(\vec{x}) \leq 0$ for all $\vec{x} \in \mathbb{R}^2 \backslash \vec{x} = 0$ (equal sign from $\dot{V}(0,x_2) = 0$).

$\therefore \dot{V}$ is negative semi-definite in the entire plane.

$\therefore V(\vec{x}(t))$ is a weak Lyapunov function by the Lyapunov stability theorem.

$\therefore$ the origin is a stable fixed point.

The conclusion we reached are confirmed by the phase portrait obtained from a numerical solution as depicted in figure 11.

As suggested by the figure the origin appears to be an asymptotically stable fixed point and we may even draw a stronger conclusion in this case. This does not follow fromt the Lyapunov stability theorem, but when we envoke the following corollary.

Corollary: *Let $V[\vec{x}(t)]$ be a weak Lyapunov function for the system $\dot{\vec{x}} = \vec{F}(\vec{x})$ in a neighbourhood of the isolated fixed point $\vec{x}_f = (0,0)$. Then if $\dot{V} \neq 0$ on any trajectory, except for the fixed point*[6]*, the origin is asymptotically stable.*

[6] Note the fixed point is a trajectory by itself.

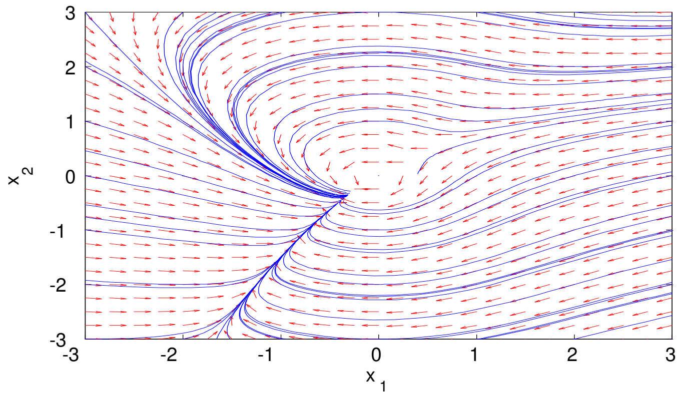

Figure 11: Phase portraits for the nonlinear system (3.57), (3.58).

Proof: We omit here the proof.

Looking again at example *iii*), we can now reach a stronger conclusion. We found that $\dot{V}(\vec{x}) = 0$ only for $\vec{x} = (0, x_2)$. This means all we need to show is that the line $(0, x_2)$ is not a trajectory. To see this we investigate the dynamical system on this line. We substitute $x_1 = 0$ in to (3.57), (3.58) and leave x_2 arbitrary. Then the system becomes $\dot{x}_1 = -x_2^2$ and $\dot{x}_2 = 0$, which means that the line is only crossed in one point and therefore $\vec{x} = (0, x_2)$ is not a trajectory for the system (3.57), (3.58). It follows therefore from the corollary that the origin is asymptotically stable.

3.5 Limit cycles

Before entering formal definitions, we start once again with a motivating example and consider the dynamical system

$$\dot{x}_1 = x_2 + x_1(1 - x_1^2 - x_2^2) \qquad (3.60)$$

$$\dot{x}_2 = -x_1 + x_2(1 - x_1^2 - x_2^2). \qquad (3.61)$$

First of all we establish that the origin is the only fixed point of the system. For this we subtract from the right hand side of (3.60) multiplied by x_2 from the right hand side of (3.61) multiplied by x_1. This simply yields $x_1^2 + x_2^2$. Setting this to zero we obtain as the determining equation for the fixed point $x_1^2 + x_2^2 = 0$. This mean that the origin is the only fixed point of the system. Next we test the linearization theorem. The Jacobian matrix at the fixed point is computed to

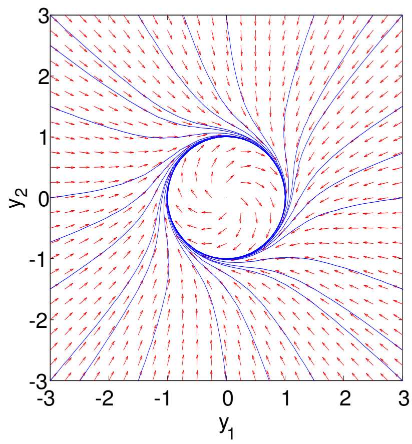

Figure 12: Phase portrait of the dynamical system (3.60), (3.61).

$$A(\vec{x}_f) = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \qquad (3.62)$$

such that the eigenvalues are $\lambda_{\pm} = 1 \pm i$. Therefore the origin is an unstable focus of the linearized system. Motivated by the fact that we have rotational symmetry we change the system to polar coordinates using the definitions (3.41)

$$\dot{x}_1 = \dot{r}\cos\vartheta - r\sin\vartheta\dot{\vartheta} = r\sin\vartheta + r\cos\vartheta(1-r^2) \tag{3.63}$$

$$\dot{x}_2 = \dot{r}\sin\vartheta + r\cos\vartheta\dot{\vartheta} = r\cos\vartheta + r\sin\vartheta(1-r^2). \tag{3.64}$$

Adding now the multiple of (3.63) with $\cos\vartheta$ and the multiple of (3.64) with $\sin\vartheta$ we obtain

$$\dot{r} = r(1-r^2). \tag{3.65}$$

Whereas subtracting the multiple of (3.64) with $\sin\vartheta$ and from (3.63) multiplied with $\cos\vartheta$ we obtain

$$\dot{\vartheta} = -1. \tag{3.66}$$

We can now distinguish three qualitatively different cases:

$r = 1$: The dynamical system (3.65) and (3.66) reduce to $\dot{r} = 0$ and $\dot{\vartheta} = -1$ in this case. This means $r = 1$ is a trajectory with constant angle velocity $\dot{\vartheta}$. We may solve (3.66) to $\vartheta(t) = -(t - t_0) + \vartheta_0$, with constants t_0, ϑ_0. Therefore $r = 1$ is a *periodic orbit* in the clockwise direction.

$r > 1$: The dynamical system (3.65) and (3.66) reduce to $\dot{r} < 0$ and $\dot{\vartheta} = -1$ in this case, which means that $r \to 1$ for $t \to \infty$, such that $\dot{r} \to 0$. There is no change in the interpretation of the angle velocity and therefore we deduce that the trajectories spiral clockwise inwards towards the periodic orbit $r = 1$.

$r < 1$: In this case the dynamical system (3.65) and (3.66) reduce to $\dot{r} > 0$ and $\dot{\vartheta} = -1$, which means $r \to 1$ for $t \to \infty$, such that $\dot{r} \to 0$. The trajectories spiral clockwise outwards towards the priodic orbit $r = 1$.

Hence all trajectories tend to $r = 1$ for $t \to \infty$. Thus it is suggestive to call $r(t) = 1$ a *stable limit cycle*.

Being now more precise, we take $\vec{x} \in \mathbb{R}^n$ and let φ_t be a flow on $\mathbb{R}^n$, that is $\varphi_t(\vec{x}(t_0)) = \vec{x}(t)$.

Definition: *The <u>ω-limit set</u> (or positive limit set) $L_\omega(\vec{x})$ of a point $\vec{x}$ contains those points which are approached by the trajectory through $\vec{x}$ as $t \to \infty$, that is*

$$L_\omega(\vec{x}) = \left\{ \vec{y} \in \mathbb{R}^n : \; \exists \text{ a sequence of times } t_n \text{ with } t_n \to \infty, \text{ such that } \lim_{n\to\infty} \varphi_{t_n}(\vec{x}) = \vec{y} \right\}$$

Definition: *The <u>α-limit set</u> (or negative limit set) $L_\alpha(\vec{x})$ of a point $\vec{x}$ contains those points which are approached by the trajectory through $\vec{x}$ as $t \to -\infty$, that is*

$$L_\alpha(\vec{x}) = \left\{ \vec{y} \in \mathbb{R}^n : \; \exists \text{ a sequence of times } t_n \text{ with } t_n \to -\infty, \text{ such that } \lim_{n\to\infty} \varphi_{t_n}(\vec{x}) = \vec{y} \right\}$$

Definition: *A closed orbit ϕ is a <u>limit cycle</u> if ϕ is a subset of an α or ω-limit set for some point $\vec{x} \notin \phi$.*

Definition: *A limit cycle ϕ is a called a* <u>*stable (unstable) limit cycle*</u>*, if $\phi = L_\omega(\vec{x})$ ($\phi = L_\alpha(\vec{x})$) for all $\vec{x}$ in some neighbourhood of the limit cycle.*
Definition: *A limit cycle ϕ is a called a* <u>*semi-stable limit cycle*</u>*, if it is a stable limit cycle for points on one side and an unstable limit cycle for point on the other side.*

Let us return to the example of the dynamical system (3.65), (3.66). Clearly the origin is a fixed point $\vec{x}_f = (0, 0)$. A limit cycle ϕ is given by the set of points for which $r = 1$. The α-limit sets and ω-limit sets are found to be

$$L_\alpha(\vec{x}) = \begin{cases} \vec{x}_f & \text{for } r < 1 \\ \varphi & \text{for } r = 1 \\ \varnothing & \text{for } r > 1 \end{cases} \qquad L_\omega(\vec{x}) = \begin{cases} \vec{x}_f & \text{for } r = 0 \\ \varphi & \text{for } r \neq 0 \end{cases} . \tag{3.67}$$

Therefore according to the previous definitions the circle ϕ of constant radius $r = 1$ is a stable limit cycle.

3.6 Poincaré-Bendixson theory

The question we address now is: What type of limit cycles can we have in general? So far we found fixed points, closed orbits and "infinity", that is empty sets. We will now show that indeed systems of the type $\dot{\vec{x}} = \vec{F}(\vec{x})$ do not possess other types of limit sets for $n = 2$, that is they do not exhibit chaotic behaviour (see later in the course what that means). This is the essense of the Poincaré-Bendixson theory.
Theorem (Poincaré-Bendixson): *Let φ_t be a flow for the system $\dot{\vec{x}} = \vec{F}(\vec{x})$ and let $\mathcal{D}$ be a closed, bounded and connected set $\mathcal{D} \in \mathbb{R}^2$, such that $\varphi_t(\mathcal{D}) \subset \mathcal{D}$ for all time. Furthermore $\mathcal{D}$ does not contain any fixed point. Then there exists at least one limit cycle in $\mathcal{D}$.*
Proof: The proof is omitted here[7].
Comment: To apply the Poincaré-Bendixson theorem we need to find a closed, bounded and connected set D, which contains no fixed point and to which all the trajectories enter but never leave. Then we can conclude that there is a limit cycle in D. Note that the validity of this theorem is restricted to $n = 2$ and in addition to continuous systems, which is implicit in the definition for the flow. This allows that chaos might exist, and indeed it does, in systems for $n > 2$ and also for discrete systems even for $n = 1$ or $n = 2$. We will encounter all these possibilities later in the course.

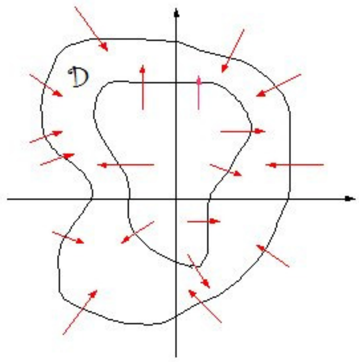

Figure 13: Set D for the Poincaré-Bendixson

3.6.1 Examples for the usage of the Poincaré-Bendixson theorem:

i) By applying the Poincaré-Bendixson theorem show that the system

$$\dot{x}_1 = x_2 + \frac{1}{4}x_1(1 - 2x_1^2 - 2x_2^2) \tag{3.68}$$

$$\dot{x}_2 = -x_1 + \frac{1}{2}x_2(1 - x_1^2 - x_2^2). \tag{3.69}$$

[7] See e.g. G.M. Zaslavsky, Physics of Chaos in Hamiltonian Systems, (World Scientific, Singapore, 1998)

has at least one periodic orbit inside the annular region $1/2 \leq r \leq 2$. Find also a smaller region containing the orbit.

Solution: Let us first establish how many fixed points we have for the above system. For this we compute $x_2 \times$(3.68)-$x_1 \times$(3.69), which yields

$$x_1^2 + x_2^2 - \frac{1}{4} x_1 x_2 = 0. \tag{3.70}$$

Let us use once more polar coordinates (3.41) and convert (3.70) into

$$r^2 - \frac{1}{4} r^2 \sin\vartheta \cos\vartheta = 0. \tag{3.71}$$

Using $2\sin\vartheta\cos\vartheta = \sin 2\vartheta$ we obtain from this for $r \neq 0$

$$\sin 2\vartheta = 8. \tag{3.72}$$

The equation (3.72) has of course no real solution for ϑ. Therefore the origin is the only fixed point of the system.

Next we compute the Jacobian matrix for $F(\vec{x})$ in (3.68), (3.69) at the fixed point

$$A = \begin{pmatrix} \frac{1}{4} & 1 \\ -1 & \frac{1}{2} \end{pmatrix}, \tag{3.73}$$

which has eigenvalues $\lambda_\pm = 3/8(1 \pm i\sqrt{7})$. This means the origin is an unstable focus.

Writing now the entire system (3.68)-(3.69) in terms of polar coordinates (3.41) converts the system into

$$\dot{r}\cos\vartheta - r\sin\vartheta\dot{\vartheta} = r\sin\vartheta + \frac{1}{4} r\cos\vartheta(1 - 2r^2) \tag{3.74}$$

$$\dot{r}\sin\vartheta + r\cos\vartheta\dot{\vartheta} = -r\cos\vartheta + \frac{1}{2} r\sin\vartheta(1 - r^2). \tag{3.75}$$

Then (3.74)$\times\cos\vartheta$+(3.75)$\times\sin\vartheta$ gives

$$\dot{r} = \frac{r}{4}(1 + \sin^2\vartheta) - \frac{r^3}{2} \tag{3.76}$$

and (3.75)$\times\cos\vartheta-$(3.74)$\times\sin\vartheta$ yields

$$\dot{\vartheta} = \frac{1}{8}\sin 2\vartheta - 1. \tag{3.77}$$

As $\dot{\vartheta} < 0$ for all (ϑ, r) there are no further fixed points apart from the origin, as we know we already concluded from (3.72). This means any annular region surrounding the origin contains no fixed points, which is an important ingredient to know for the application of the Poincaré-Bendixson theorem. Let us next look search for the existence of limit cycles and periodic orbits. Consider for this the closed, bounded and connected domain

$$\mathcal{D} = \left\{ (r, \vartheta) : \frac{1}{2} \leq r \leq 2 \right\}, \tag{3.78}$$

as specified in the task. Since $\dot{\vartheta} \neq 0 \ \forall (\vartheta, r) \in \mathcal{D}$ there are no fixed points in $\mathcal{D}$. At the boundaries of $\mathcal{D}$ we compute

$$r = \frac{1}{2}: \qquad \dot{r} = \frac{1}{8}(1 + \sin^2 \vartheta) - \frac{1}{16} = \frac{1}{16}(1 + 2\sin^2 \vartheta) > 0 \quad \forall \vartheta \tag{3.79}$$

$$r = 2: \qquad \dot{r} = \frac{1}{2}(1 + \sin^2 \vartheta) - 4 < 0 \quad \forall \vartheta. \tag{3.80}$$

Therefore, we conclude that trajectories which enter the closed, bounded and connected set $\mathcal{D}$, do not leave it anymore. Thus by the Poincaré-Bendixson theorem we deduce that there is at least one limit cycle in $\mathcal{D}$.

Next we determine a smaller annular region with this property: Since we know that $\dot{\vartheta} \neq 0 \ \forall (\vartheta, r)$ except for the origin, there is no problem with the fixed points. On the inner boundary we require that $\dot{r} > 0 \ \forall \vartheta$, which is equivalent to

$$\frac{1}{4} r(1 + \sin^2 \vartheta) - \frac{1}{2} r^3 > 0 \qquad \forall \vartheta. \tag{3.81}$$

This means that for $r \neq 0$ we demand

$$r^2 < \frac{1}{2}(1 + \sin^2 \vartheta) \quad \forall \vartheta. \tag{3.82}$$

In figure 15 we have plotted the right hand side of equation (3.82) as a function of ϑ. In order to obtain an annular region we would now like to eliminate the ϑ-dependence in this estimate and substitute the right hand side simply by a real number. Clearly if we replace $\frac{1}{2}(1 + \sin^2 \vartheta)$ by its minimum the inequality will hold still hold for all values of ϑ. (See also figure for this.) Computing the minimum of the upper bound then yields

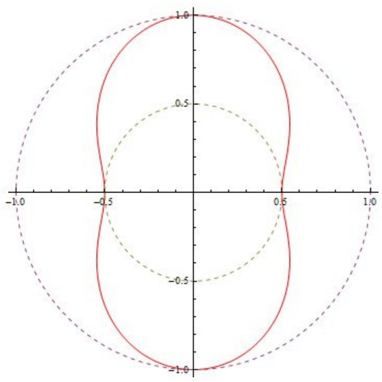

Figure 14: Radial bounds for the function $\frac{1}{2}(1 + \sin^2 \vartheta)$

$$r^2 < \frac{1}{2} \min(1 + \sin^2 \vartheta) = \frac{1}{2} \quad \forall \vartheta. \tag{3.83}$$

On the outer boundary we require that $\dot{r} < 0 \ \forall \vartheta$, which is equivalent to

$$\frac{1}{4} r(1 + \sin^2 \vartheta) - \frac{1}{2} r^3 < 0 \qquad \forall \vartheta. \tag{3.84}$$

This means that for $r \neq 0$ we have to have

$$r^2 > \frac{1}{2}(1 + \sin^2 \vartheta) \quad \forall \vartheta. \tag{3.85}$$

Now we are investigating the outer boundary, therefore we will not destroy the bound in (3.85) by making the lower bound greater and replace it by its maximal value. We compute for this the maximum of the lower bound, such that

$$r^2 > \frac{1}{2} \max(1 + \sin^2 \vartheta) = 1 \quad \forall \vartheta. \tag{3.86}$$

This means we can now define a new closed, bounded and connected set

$$\mathcal{D}^{\varepsilon} = \left\{ (r, \vartheta) : \frac{1}{\sqrt{2}} - \varepsilon \leq r \leq 1 + \varepsilon \right\}, \tag{3.87}$$

where $0 < \varepsilon \ll 1$. We need to be careful here and introduce the ε as in our above estimates we did not include the equal signs, which is necessary for the set to be closed, bounded and connected. For the Poincaré-Bendixson theorem to be applicable we note first that there are obviously no fixed points in $\mathcal{D}^{\varepsilon}$. Since $\dot{r} > 0$ on the inner boundary and $\dot{r} < 0$ on the outer boundary, this means that trajectories which enter the domain $\mathcal{D}^{\varepsilon}$ do not leave it anymore. This implies by the Poincaré-Bendixson theorem that there is at least one periodic orbit in $\mathcal{D}^{\varepsilon}$. Since $r = 1/\sqrt{2}$ and $r = 1$ are no trajectories of the system, the above statements are also true for $\varepsilon = 0$, that is we may consider the new optimized domain

$$\mathcal{D} = \left\{ (r, \vartheta) : \frac{1}{\sqrt{2}} \leq r \leq 1 \right\} \tag{3.88}$$

and conclude that the system (3.68)-(3.69) possesses for sure a limit cycle in $\mathcal{D}$.

ii) Show that the second order differential equation

$$\ddot{x} - (1 - 3x^2 - 2\dot{x})\dot{x} + x = 0 \tag{3.89}$$

has at least one periodic solution.

Solution: Using once again the transformation (3.2) introduced earlier we convert this second order differential equation first into two coupled first order differential equations. We obtain

$$\dot{x}_1 = x_2, \tag{3.90}$$

$$\dot{x}_2 = -x_1 + x_2(1 - 3x_1^2 - 2x_2^2). \tag{3.91}$$

Clearly $\vec{x}_f = (0, 0)$ is the only fixed point of the system. The Jacobian matrix at the fixed point is computed to

$$A(\vec{x}_f) = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, \tag{3.92}$$

such that the eigenvalues are $\lambda_{\pm} = 1/2 \pm i\sqrt{3}/2$. Therefore the origin is an unstable focus of the linearized system. Next we convert the system (3.90), (3.91) to polar coordinates (3.41), which gives

$$\dot{r}\cos\vartheta - r\sin\vartheta\dot{\vartheta} = r\sin\vartheta \tag{3.93}$$

$$\dot{r}\sin\vartheta + r\cos\vartheta\dot{\vartheta} = -r\cos\vartheta + r\sin\vartheta(1 - r^2\cos^2\vartheta - 2r^2). \tag{3.94}$$

Then (3.93)$\times \cos\vartheta$+(3.94)$\times \sin\vartheta$ yields

$$\dot{r} = r\sin^2\vartheta(1 - r^2\cos^2\vartheta - 2r^2). \tag{3.95}$$

In order to find the limit cycle domain we require $\dot{r} > 0 \ \forall\vartheta$ on the inner boundary, which is equivalent to

$$1 - r^2(2 + \cos^2\vartheta) > 0 \qquad \forall\vartheta. \tag{3.96}$$

This means that for $r \neq 0$ we demand

$$r^2 < \frac{1}{2+\cos^2\vartheta} \qquad \forall \vartheta. \tag{3.97}$$

We now use the same logic as in the previous example and replace the bound on the right hand of (3.97) by its minimum, as this guarantee the validity for all values of ϑ. We compute the minimum of the bound to

$$r^2 < \min\left[\frac{1}{(2+\cos^2\vartheta)}\right] = \frac{1}{3} \quad \forall \vartheta. \tag{3.98}$$

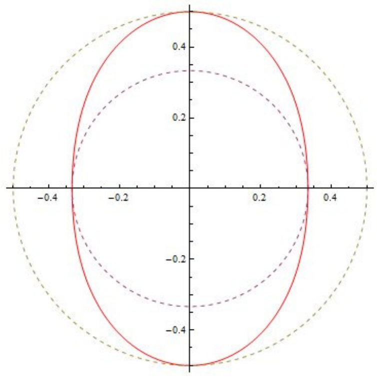

Figure 15: Radial bounds for the function $1/(2+\cos^2\vartheta)$

On the outer boundary we need $\dot{r} < 0 \; \forall \; \vartheta$, which is equivalent to

$$1 - r^2(2+\cos^2\vartheta) < 0 \qquad \forall \vartheta. \tag{3.99}$$

This means that for $r \neq 0$ we require

$$r^2 > \frac{1}{(2+\cos^2\vartheta)} \qquad \forall \vartheta. \tag{3.100}$$

Now we are investigating the outer boundary, therefore we will not invalidate the estimate by increasing the lower bound greater. We compute for this the maximum of the bound

$$r^2 > \max\left[\frac{1}{(2+\cos^2\vartheta)}\right] = \frac{1}{2} \qquad \forall \vartheta. \tag{3.101}$$

This means we can now define a new domain

$$\mathcal{D}^\varepsilon = \left\{(r,\vartheta) : \frac{1}{\sqrt{3}} - \varepsilon \leq r \leq \frac{1}{\sqrt{2}} + \varepsilon\right\}, \tag{3.102}$$

where $0 < \varepsilon \ll 1$. We introduced here the ε for the same reason as in the previous example. For the Poincaré-Bendixson theorem to be applicable we note first that there are obviously no fixed points in $\mathcal{D}^\varepsilon$. Since $\dot{r} > 0$ on the inner boundary and $\dot{r} < 0$ on the outer boundary, this means that trajectories which enter the domain $\mathcal{D}^\varepsilon$ do not leave it anymore. This implies by the Poincaré-Bendixson theorem that there is at least one periodic orbit in $\mathcal{D}^\varepsilon$. Since $r = 1/\sqrt{2}$ and $r = 1/\sqrt{3}$ are no trajectories of the system, the above statements are also true for $\varepsilon = 0$, that is we may consider the new optimized domain

$$\mathcal{D} = \left\{(r,\vartheta) : \frac{1}{\sqrt{3}} \leq r \leq \frac{1}{\sqrt{2}}\right\}, \tag{3.103}$$

and deduce that the system (3.90)-(3.91) has a limt cycle in there. In turn this means that the differential equation (3.89) has a periodic solution in $\mathcal{D}$.

The next theorem is a further necessary condition for a limit cycle to exist.
Theorem 4: *A limit cycle contains at least one fixed point.*

Proof: Omitted.

We also illustrate the working of this theorem with an example.
Example: We show that following system has no limit cycle

$$\dot{x}_1 = 1 + x_2^2 - \exp(x_1 x_2) \tag{3.104}$$

$$\dot{x}_2 = x_1 x_2 + 5. \tag{3.105}$$

In order to find a fixed point we need to solve

$$1 + x_2^2 = \exp(x_1 x_2) \qquad \text{and} \qquad x_1 x_2 + 5 = 0. \tag{3.106}$$

There is obviously no real solution for (3.106), which mean the system (3.104), (3.105) has no fixed points. Therefore according to theorem 5 the system can also not have any limit cycles.
Being now equipped with tools, which allows us to determine that a certain region has to contain a limit cycle, we will discuss next a criterium which permits us to exclude this possibility.

3.6.2 Bendixson's criterium

The following criterium is a sufficient condition for the non-existence of limit cycles.
Theorem (Bendixson's criterium): *Let $\mathcal{D}$ be a simply connected region (this means "there are no holes in $\mathcal{D}$") of the phase plane in which the function $\vec{F}(\vec{x})$ of the system $\dot{\vec{x}} = \vec{F}(\vec{x})$ has the property that its divergence is of constant sign, i.e.*

$$\operatorname{div} \vec{F} = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} < 0 \qquad \text{or} \qquad \operatorname{div} \vec{F} = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} > 0. \tag{3.107}$$

Then the system posesses no closed orbit contained entirely in $\mathcal{D}$.

Proof: We use Green's theorem in the plane. Take $P(x_1, x_2)$ and $Q(x_1, x_2)$ to be real valued functions and $\mathcal{C}$ a curve enclosing the simply connected region $\mathcal{D}$ in the positive mathematical sense. Then Green's theorem states

$$\oint_{\mathcal{C}} [P(x_1, x_2) dx_1 + Q(x_1, x_2) dx_2] = \iint_{\mathcal{D}} \left[\frac{\partial Q(x_1, x_2)}{\partial x_1} - \frac{\partial P(x_1, x_2)}{\partial x_2} \right] dx_1 dx_2. \tag{3.108}$$

We take now $P(x_1, x_2) = -F_2(x_1, x_2)$ and $Q(x_1, x_2) = F_1(x_1, x_2)$. Furthermore, we assume that $\mathcal{C}$ is a limit cycle of period T. Then the left hand side of (3.108) yields

$$\oint_{\mathcal{C}} (F_1 dx_2 - F_2 dx_1) = \int_0^T dt (F_1 \dot{x}_2 - F_2 \dot{x}_1) = \int_0^T dt (\dot{x}_1 \dot{x}_2 - \dot{x}_2 \dot{x}_1) = 0. \tag{3.109}$$

The right hand side is, however, never vanishing

$$\iint_{\mathcal{D}} \left(\frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} \right) dx_1 dx_2 = \iint_{\mathcal{D}} \operatorname{div} \vec{F} \; dx_1 dx_2 \neq 0 \tag{3.110}$$

as the integrant does not change sign. Therefore $\mathcal{C}$ can not be a limit cycle.$\square$

Let us illustrate the use of Bendixson's criterium with some examples.

Example 1: Consider the dynamical system

$$\dot{x}_1 = x_2 + x_1(1 - x_1^2 - x_2^2) = F_1(x_1, x_2) \tag{3.111}$$

$$\dot{x}_2 = x_1 + x_2(1 - x_1^2 - x_2^2) = F_2(x_1, x_2). \tag{3.112}$$

Computing the divergence for this system gives

$$\operatorname{div} \vec{F} = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} = (1 - 3x_1^2 - x_2^2) + (1 - x_1^2 - 3x_2^2) = 2 - 4(x_1^2 + x_2^2), \tag{3.113}$$

which means

$$\operatorname{div} \vec{F} = \begin{cases} < 0 & \text{for} \quad r^2 > 1/2 \\ > 0 & \text{for} \quad r^2 < 1/2 \end{cases}. \tag{3.114}$$

Therefore we conclude from Bendixson's criterium that there is no periodic orbit entirely contained inside the simply connected region

$$\mathcal{D} = \left\{ (x_1, x_2) : x_1^2 + x_2^2 < \frac{1}{2} \right\}. \tag{3.115}$$

Hence, if a closed orbit exists it has to be entirely in the complement of $\mathcal{D}$, that is in

$$\mathcal{D}^\vee = \left\{ (x_1, x_2) : x_1^2 + x_2^2 > \frac{1}{2} \right\} \tag{3.116}$$

or intersect the circle $x_1^2 + x_2^2 = 1/2$. Note that $\mathcal{D}^\vee$ despite the fact that $\operatorname{div} \vec{F} > 0$ in $\mathcal{D}^\vee$, i.e. the divergence has constant sign, we can not exclude the possibility of the existence of a limit cycle in $\mathcal{D}^\vee$ since it is not a simply connected region and therefore Bendixson's criterium can not be applied in this case.

Example 2: In this example we apply Bendixson's criterium and an additional argument. We consider the system

$$\dot{x}_1 = 4x_1 - 2x_1^2 - x_2^2, \tag{3.117}$$

$$\dot{x}_2 = x_1 + x_2 x_1^2. \tag{3.118}$$

such that the divergence results to

$$\operatorname{div} \vec{F} = 4 - 4x_1 + x_1^2 = (x_1 - 2)^2 > 0 \qquad \text{for} \quad x_1 \neq 2. \tag{3.119}$$

According to the Bendixson's criterium we can not have a limit cycle entirely contained in either of the two half planes $x_1 < 2$ or $x_1 > 2$. However, according to Bendixson's criterium a closed orbit which crosses the line $x_1 = 2$ could still be a possibility. Nonetheless, on this line we have $\dot{x}_1 = -x_2^2 \leq 0$, which means the line is always crossed from the right to the left. Thus it is not possible to have a closed orbit crossing this line and therefore the system can not possess any limit cycle.

Example 3: Let us now discuss an example in which we have to make use of all the above criteria and consider the system

$$\dot{x}_1 = x_1^2 - x_2 - 1, \tag{3.120}$$
$$\dot{x}_2 = x_1 x_2 - 2x_2. \tag{3.121}$$

First we apply Bendixson's criterium and compute for this purpose the divergence

$$\operatorname{div} \vec{F} = 2x_1 + x_1 - 2 = 3x_1 - 2 = \begin{cases} < 0 & \text{for } x_1 < 2/3 \\ = 0 & \text{for } x_1 = 2/3 \\ > 0 & \text{for } x_1 > 2/3 \end{cases}. \tag{3.122}$$

According to Bendixson's criterium we can not have a limit cycle entirely contained in either of the two half planes $x_1 < 2/3$ or $x_1 > 2/3$. However, a closed orbit which crosses the line $x_1 = 2/3$ constitutes still a possibility. On this line we have $\dot{x}_1 = -5/9 - x_2$, such that $\dot{x}_1 < 0$ for $x_2 > -5/9$ and $\dot{x}_1 > 0$ for $x_2 < -5/9$. This means for $x_2 > -5/9$ trajectories cross this line from the right to the left and for $x_2 < -5/9$ from the left to the right. Thus unlike as in the previous example we can still construct a closed orbit crossing this line and the possibility of a limit cycle can not yet be excluded. Let us therefore also invoke theorem 4 and compute for this purpose the fixed points of the system. Solving

$$x_1^2 - x_2 - 1 = 0 \qquad \text{and} \qquad x_1 x_2 - 2x_2 = 0, \tag{3.123}$$

we find the three fixed points $x_f^{(1)} = (1,0)$, $x_f^{(2)} = (-1,0)$ and $x_f^{(3)} = (2,3)$. The possibility to form a closed orbit encircling one of these fixed points and crossing the line $x_1 = 2/3$ still exists. Next we consider the line $x_2 = 0$ on which $\dot{x}_2 = 0$ and $\dot{x}_1 = x_1^2 - 1$. From this equation we see that we can not cross this line without being dragged into $x_f^{(2)}$ for $x_1 < 1$ or to be repelled to positive infinity for $x_1 > 1$. Thus all possibilities have been exhausted and we can not draw any closed orbit surrounding one of the fixed points and at the same time crossing the line $x_1 = 2/3$. We therefore conclude that the system can not possess any limit cycle.

3.7 Bifurcation theory

We will now consider systems similar to those before, but with the difference that the vector function $\vec{F}$ depends in addition on a real parameter λ. We consider first systems in one dimension

$$\dot{x} = F(x, \lambda) \qquad\qquad \lambda \in \mathbb{R}. \tag{3.124}$$

Bifurcation theory investigates how the number of steady solutions of systems of the type (3.124) depend on the parameter λ. A *bifurcation* occurs if the solution of (3.124) changes its qualitative behaviour as the parameter λ varies. Considering $F(x, \lambda) = 0$ leads to a plot in the (x, λ)-plane called the *bifurcation diagram.* As usual we commence by studying a motivating example for a bifurcation.

Example: Consider the dynamical system

$$\dot{x} = \lambda - x^2 = F(x, \lambda). \tag{3.125}$$

This means for the fixed points $\dot{x} = 0$ we have

$$F(x, \lambda) = 0 \qquad \Leftrightarrow \qquad x^2 = \lambda \qquad \Leftrightarrow \qquad x = \pm\sqrt{\lambda}. \tag{3.126}$$

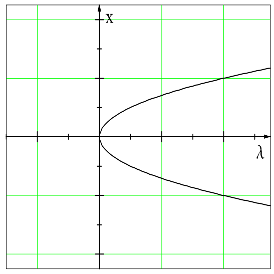

Figure 16: Bifurcation diagram.

The bifurcation diagram can then be sketched as in figure 16, i.e we plot x as a function of λ. The curve $F(x, \lambda) = 0$ is called the *equilibrium curve* for the dynamical system (3.125). Similarly as for points, a bifurcation diagram, being just a collection of points, also contains information on the stability of the fixed points. In the example (3.125) we find the following for the time derivative of x

$$\begin{array}{lllll} \text{for} & \lambda < 0 & & \Rightarrow & \dot{x} < 0 \;\; \forall x, \\ \text{for} & \lambda > 0 & \wedge \quad -\sqrt{\lambda} < x < \sqrt{\lambda} & \Rightarrow & \dot{x} > 0, \\ \text{for} & \lambda > 0 & \wedge \quad x > \sqrt{\lambda} & \Rightarrow & \dot{x} < 0, \\ \text{for} & \lambda > 0 & \wedge \quad x < -\sqrt{\lambda} & \Rightarrow & \dot{x} < 0. \end{array}$$

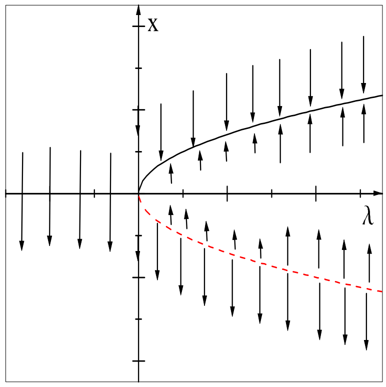

Figure 17: Bifurcation diagram for the system (3.125).

Elaborating further on the figure 16, this information is encoded into the bifurcation diagram as depicted in figure 17.
We have used here the following conventions:

- Unstable solutions are indicated by dashed lines.

- Stable solutions are continuous (solid) lines.

- As before we indicated the directions of the evolution by an arrow.

3.7.1 Different types of bifurcations

Bifurcation diagrams may exhibit quite different types of qualitative behaviour at points where bifurcations occur. One organizes them by catorizing according to different types of derivatives at these points and by grouping them under different names.

Turning point:
The names given are related to the overall shape of the bifurcation. The bifurcation point at origin in figure 17 is called a turning point for obvious reasons. More precisely we can capture the occurrence of such type of behaviour in the following definition.
Definition: *Let $\lambda(x)$ be the solution of the equation $F(x, \lambda) = 0$, that is $\lambda(x)$ is an equilibrium curve parameterizing the fixed points for the system $\dot{x} = F(x, \lambda)$. A specific point (x_0, λ_0) on this curve is called a* <u>turning point</u> *if $\partial F/\partial\lambda|_{(x_0,\lambda_0)} \neq 0$ and $\partial\lambda/\partial x$ changes sign at this point.*

Example: Consider once more the system (3.125). Obviously $\lambda(x) = x^2$ and the fixed points and we therefore compute

$$\frac{\partial F}{\partial \lambda} = 1 \qquad \text{and} \qquad \frac{\partial \lambda}{\partial x} = \frac{\partial (x^2)}{\partial x} = 2x. \tag{3.127}$$

Since $\partial\lambda/\partial x$ changes sign at $x = 0$ it follows that $\lambda_0 = 0$ and therefore $x_0 = 0$. This means according to the definition $(x_0, \lambda_0) = (0, 0)$ is a turning point.

Transcritical bifurcation:
A different type of behaviour arises when the partial differentials of F vanish at the bifurcation point.
Definition: *Let $\lambda(x)$ be an equilibrium curve for the system $\dot{x} = F(x, \lambda)$. A point (x_0, λ_0) on this curve is called a transcritical bifurcation if $\partial F/\partial\lambda|_{(x_0,\lambda_0)} = 0$, $\partial F/\partial x|_{(x_0,\lambda_0)} = 0$ and in addition two and only two branches of the equilibrium curve pass through this point which have both distinct tangents at (x_0, λ_0).*

Example: The following system possesses a transcritical bifurcation point

$$\dot{x} = \lambda x - \gamma x^2 = F(x, \lambda), \qquad \gamma > 0, \lambda \equiv \text{bifurcation parameter}. \tag{3.128}$$

Note that when we have two or more parameters in our equations we have to be clear which one is taken to be the bifurcation parameter. The fixed points are easily identified to be at the two lines $x = 0$ and $x = \lambda/\gamma$. We compute for (3.128)

$$\frac{\partial F}{\partial x} = \lambda - 2\gamma x \qquad \text{and} \qquad \frac{\partial F}{\partial \lambda} = x. \tag{3.129}$$

Setting both equations to zero we obtain the point $(x_0, \lambda_0) = (0, 0)$ as a potential candidate for a transcritical bifurcation. In order to verify the second part of the definition let us plot the bifurcation diagram. First we have to analyze the direction of evolution. We find

$$
\begin{array}{lllll}
\text{for} & \lambda < 0 & \wedge & x > 0 & \Rightarrow \; \dot{x} < 0, \\
\text{for} & \lambda > 0 & \wedge & x < 0 & \Rightarrow \; \dot{x} < 0, \\
\text{for} & \lambda > 0 & \wedge & 0 < x < \lambda/\gamma & \Rightarrow \; \dot{x} > 0, \\
\text{for} & \lambda > 0 & \wedge & x > \lambda/\gamma & \Rightarrow \; \dot{x} < 0, \\
\text{for} & \lambda < 0 & \wedge & x < \lambda/\gamma & \Rightarrow \; \dot{x} < 0, \\
\text{for} & \lambda < 0 & \wedge & \lambda/\gamma < x < 0 & \Rightarrow \; \dot{x} > 0.
\end{array}
$$

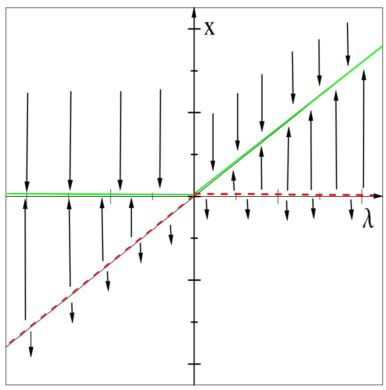

Figure 18: Bifurcation diagram for the system (3.128).

Encoding this information into the bifurcation diagram we can draw figure 18 by using the conventions as stated above. This means that at the point $(x_0, \lambda_0) = (0, 0)$ we have a transcritical bifurcation as two and two two branches of the equilibrium curve pass through this point, namely the lines $x = 0$ and $x = \lambda/\gamma$. Clearly tangents at (x_0, λ_0) to both of these lines are different.

Pitchfork bifurcation:

A further distinct type of bifurcation emerges when in addition to the above $d\lambda/dx$ changes sign on one of the branches. The origin for the name pitchfork bifurcation, given to these type of bifurcation, is apparent when looking at the shape of the equlibrium curve, as for instance illustrated in 19.

Definition: *Let $\lambda(x)$ be an equilibrium curve for the system $\dot{x} = F(x, \lambda)$. A point (x_0, λ_0) on this curve is called a a pitchfork bifurcation if $\partial F/\partial\lambda|_{(x_0,\lambda_0)} = 0$, $\partial F/\partial x|_{(x_0,\lambda_0)} = 0$ and $d\lambda/dx$ changes sign on one branch of the equilibrium curve with distinct tangents.*

Example: The following system possesses a pitchfork bifurcation point

$$\dot{x} = \lambda x - \gamma x^3 = F(x, \lambda), \qquad \gamma > 0, \lambda \equiv \text{bifurcation parameter.} \tag{3.130}$$

The fixed points are found to be at the three lines $x = 0$ and $x = \pm\sqrt{\lambda/\gamma}$ for $\lambda > 0$. We compute now for the function F in (3.130)

$$\frac{\partial F}{\partial x} = \lambda - 3\gamma x^2 \qquad \text{and} \qquad \frac{\partial F}{\partial \lambda} = x. \tag{3.131}$$

Setting both equations to zero we obtain identify $(x_0, \lambda_0) = (0, 0)$ as a possible pitchfork bifurcation point. Once again, in order to see the remaining part of the definition we plot the bifurcation diagram. We find

$$\begin{array}{llll}
\text{for } \lambda < 0 & \wedge & x > 0 & \Rightarrow \dot{x} < 0, \\
\text{for } \lambda < 0 & \wedge & x < 0 & \Rightarrow \dot{x} > 0, \\
\text{for } \lambda > 0 & \wedge & x > \sqrt{\lambda/\gamma} & \Rightarrow \dot{x} < 0, \\
\text{for } \lambda > 0 & \wedge & 0 < x < \sqrt{\lambda/\gamma} & \Rightarrow \dot{x} > 0, \\
\text{for } \lambda > 0 & \wedge & -\sqrt{\lambda/\gamma} < x < 0 & \Rightarrow \dot{x} < 0, \\
\text{for } \lambda > 0 & \wedge & x < -\sqrt{\lambda/\gamma} & \Rightarrow \dot{x} > 0.
\end{array}$$

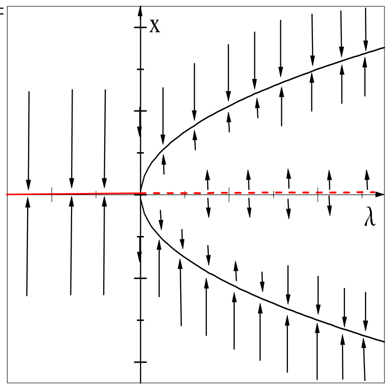

Figure 19: Bifurcation diagram for the system (3.130).

Next we compute the change of λ with respect to x on the branches $x = \pm\sqrt{\lambda/\gamma}$

$$\frac{d\lambda}{dx} = \frac{d(\gamma x^2)}{dx} = 2\gamma x. \tag{3.132}$$

Clearly $\frac{d\lambda}{dx}$ changes sign at $x = 0$. Therefore we have a pitchfork bifurcation at $(x_0, \lambda_0) = (0, 0)$, which we depict in figure 19 by using the above conventions.

Pitchfork bifurcations are further classsified according to the direction into which the fork points. More precisely:

Definition: *When the bifurcated solution arises as λ increases (decreases) above the values for which the bifucation occurs, the pitchfork bifurcation is called supercritical (subcritical).*

According to these definitions the system (3.130) constitutes a supercritical bifurcation. It is easily seen (exercise) that taking $\gamma < 0$ will give a subcritical bifurcation. The corresponding bifucation diagram is 19 reflected about the x-axis and with all arrows reversed.

Hopf bifurcation:

Having classified various bifurcations in one dimension let us return to two dimensional systems, but now with the difference that we have in addition a bifurcation parameter λ at our disposal which we shall vary

$$\dot{x}_1 = F_1(x_1, x_2, \lambda), \tag{3.133}$$

$$\dot{x}_2 = F_2(x_{1,}x_2, \lambda). \tag{3.134}$$

As usual let us start by investigating a motivating example.

Example: Consider the dynamical system

$$\dot{x}_1 = \lambda x_1 - x_2 - x_1(x_1^2 + x_2^2), \tag{3.135}$$

$$\dot{x}_2 = x_1 + \lambda x_2 - x_2(x_1^2 + x_2^2), \tag{3.136}$$

with $\lambda \in \mathbb{R}$ taken to be a bifurcation parameter. It easy to see that the fixed point of the system is $\vec{x}_f = (0,0)$. The Jacobian matrix of the linearized system at the fixed point is computed to

$$A(\vec{x}_f) = \begin{pmatrix} \lambda & -1 \\ 1 & \lambda \end{pmatrix}, \tag{3.137}$$

such that the eigenvalues are $e_\pm = \lambda \pm i$. Depending on the values for λ we have now the following possibilities:

$\lambda < 0$: the eigenvalues are complex with negative real part $\Rightarrow \vec{x}_f \equiv$ stable focus

$\lambda = 0$: the eigenvalues are purely imaginary $\Rightarrow \vec{x}_f \equiv$ centre

$\lambda > 0$: the eigenvalues are complex with positive real part $\Rightarrow \vec{x}_f \equiv$ unstable focus.

Let us confirm this behaviour by looking in more detail at the system. We convert for this purpose the equations (3.135), (3.136) to polar coordinates (3.41), which gives

$$\dot{r}\cos\vartheta - r\sin\vartheta\dot{\vartheta} = \lambda r\cos\vartheta - r\sin\vartheta - r\cos\vartheta r^2, \tag{3.138}$$

$$\dot{r}\sin\vartheta + r\cos\vartheta\dot{\vartheta} = r\cos\vartheta + \lambda r\sin\vartheta - r\sin\vartheta r^2. \tag{3.139}$$

Then (3.138)$\times\cos\vartheta+$(3.139)$\times\sin\vartheta$ yields

$$\dot{r} = r(\lambda - r^2), \tag{3.140}$$

and (3.138)$\times\sin\vartheta-$(3.139)$\times\cos\vartheta$ gives

$$\dot{\vartheta} = 1. \tag{3.141}$$

We deduce from this that for $\lambda > 0$ the circle $r = \sqrt{\lambda}$ is a closed orbit. In fact this is a stable limit cycle as $\dot{r} < 0$ for $r > \sqrt{\lambda}$ and $\dot{r} > 0$ for $r < \sqrt{\lambda}$. For $\lambda < 0$ we have instead always $\dot{r} < 0$ such that all trajectories spiral anti-clockwise towards the origin. Such type of bifurcations for which the fixed point at the origin changes its characteristic as the bifurcation parameter varies, i.e. from stable to a centre and then to an unstable fixed point are called *Hopf bifurcations*. Let us study these type of behaviour more formally.

3.7.2 The Hopf bifurcation theorem

The next theorem is a criterium which provides us with tools to decide whether such type of bifurcations may occur.

Theorem (Hopf bifurcation theorem): *Let the point (0,0,λ), with $\lambda \in \mathbb{R}$, be a fixed point for the system*

$$\dot{x}_1 = F_1(x_1, x_2, \lambda), \tag{3.142}$$

$$\dot{x}_2 = F_2(x_1, x_2, \lambda), \tag{3.143}$$

for all values of λ. If for a particular value of λ, say $\lambda = \tilde{\lambda}$,

i) *the eigenvalues $e_1(\lambda)$ and $e_2(\lambda)$ of the linearized system are purely imaginary, i.e. $e_1(\tilde{\lambda}) \in iR$ and $e_2(\tilde{\lambda}) \in iR$,*

ii) *the real part of the eigenvalues* $\mathrm{Re}(e_1(\lambda)) = \mathrm{Re}(e_2(\lambda))$ *satisfies*

$$\frac{d}{d\lambda} \mathrm{Re}(e_{1/2}(\lambda)) \bigg|_{\lambda = \tilde{\lambda}} > 0, \tag{3.144}$$

iii) *the origin is asymptotically stable for $\lambda = \tilde{\lambda}$,*

then the following statements hold:

a) *The point with $\lambda = \tilde{\lambda}$ is a bifurcation point of the system.*

b) *For $\lambda \in (\lambda_1, \tilde{\lambda})$ with some $\lambda_1 < \tilde{\lambda}$ the origin is a stable focus.*

c) *For $\lambda \in (\tilde{\lambda}, \lambda_2)$ with some $\lambda_2 > \tilde{\lambda}$ the origin is an unstable focus surrounded by a stable limit cycle whose size increases with λ.*

Proof: Omitted here[8].

Note that in order to apply the theorem we have to show in condition iii) that the origin is asymptotically stable for $\lambda = \tilde{\lambda}$. Unfortunately for the case at hand we can not obtain any information about the origin from the linearization theorem as the linearization is a centre. The alternative method we have learned so far is to search for Lyapunov functions, which is, however, often very difficult. Therefore we introduce here yet another approach which makes use of the so-called *stability index* and will allow us to draw the desired conclusion.

Corollary: *Suppose that for the system*

$$\dot{\vec{x}} = \vec{F}(\vec{x}), \tag{3.145}$$

we have transformed the linearized system

$$\dot{\vec{x}} = A\vec{x}, \tag{3.146}$$

[8] See for instance J.E. Marsden and M. MacCracken, The Hopf Bifurcation and its Application (Appl. Math. Sciences, Vol 19, Springer Ney York, 1976)

with the help of $\vec{x} = U\vec{y}$ into the Jordan normal form

$$\dot{\vec{y}} = U^{-1}AU\vec{y} = J\vec{y} = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}, \tag{3.147}$$

with $\omega \in \mathbb{R}^+$. Accordingly the entire system (3.145) is transformed with the help of $\vec{x} = U\vec{y}$ to

$$\dot{\vec{y}} = \vec{Y}(\vec{y}). \tag{3.148}$$

Then if the stability index

$$I = \omega\left(Y^1_{111} + Y^1_{122} + Y^2_{112} + Y^2_{222}\right) + Y^1_{11}(Y^2_{11} - Y^1_{12}) + Y^2_{22}(Y^2_{12} - Y^1_{22}) + Y^2_{11}Y^2_{12} - Y^1_{22}Y^1_{12}$$

computed from (3.148) is negative, the origin is asymptotically stable. We have used here the abbreviations

$$Y^i_{jk} = \left.\frac{\partial^2 Y_i}{\partial y_j \partial y_k}\right|_{(0,0)} \qquad \text{and} \qquad Y^i_{jkl} = \left.\frac{\partial^3 Y_i}{\partial y_j \partial y_k \partial y_l}\right|_{(0,0)}. \tag{3.149}$$

Proof: Omitted here[9].

Let us illustrate with some examples how this theorem can be applied.

Example 1: Investigate whether the dynamical system

$$\dot{x}_1 = 7x_2, \tag{3.150}$$
$$\dot{x}_2 = -(x_1^2 - \lambda)x_2 - 7x_1 - 2x_1^3, \tag{3.151}$$

possess a Hopf bifurcation point at the origin when the parameter λ is varied.

Solution: We start by computing the Jacobian matrix at the origin to

$$A(\vec{x}_f = (0,0)) = \begin{pmatrix} 0 & 7 \\ -7 & \lambda \end{pmatrix}.$$

The two eigenvalues are computed to

$$\det A_e = e^2 - \lambda e + 49 = 0 \quad \Rightarrow \quad e_\pm = \frac{\lambda}{2} \pm \frac{1}{2}\sqrt{\lambda^2 - 196}. \tag{3.152}$$

The Hopf bifurcation theorem requires that:

i) For a particular value of λ the eigenvalues are purely imaginary. This is indeed the case for $\lambda = \tilde{\lambda} = 0$ we have $e_\pm = \pm i\sqrt{7}$.

ii) We compute

$$\left.\frac{d}{d\lambda}\operatorname{Re}(e_\pm(\lambda))\right|_{\lambda=\tilde{\lambda}=0} = \frac{1}{2} > 0. \tag{3.153}$$

[9] See for instance J.E. Marsden and M. MacCracken, The Hopf Bifurcation and its Application (Appl. Math. Sciences, Vol 19, Springer Ney York, 1976)

iii) We have to show that the origin is asymptotically stable, which we do by means of the corollary. Note that the system is already in Jordan normal form, such that $A = J$ and $\vec{X} = \vec{Y}$. This means we can compute I directly. Considering (3.150) and (3.151) we observe that the only nonvanishing term in the definition for the stability index is $Y^2_{112} = -2$. Together with $\omega = 7$ we compute therefore

$$I = \omega Y^2_{112} = -14.$$

As I is negative it follows therefore from the corollary that the origin is asymptotically stable.

Thus the Hopf bifurcation theorem applies, which means in particular that $(0, 0, 0)$ is a bifurcation point for the system (3.150), (3.151).

Example 2: Show that the second order differential equation

$$\ddot{x} + (x^2 - \lambda)\dot{x} + 2x + x^3 + \alpha\dot{x}^3 = 0 \qquad \alpha, \lambda \in \mathbb{R}, \tag{3.154}$$

has a bifurcation point at $\lambda = 0$ when $\alpha = 1$ and that in this case the solution to (3.154) is oscillatory for some $\lambda > 0$. Can you draw the same conclusion for $\alpha = -1$?

Solution: Using once more the transformation (3.2), we convert the equation (3.154) into two first order equations

$$\dot{x}_1 = \dot{x} = x_2, \tag{3.155}$$
$$\dot{x}_2 = \ddot{x} = -(x_1^2 - \lambda)x_2 - 2x_1 - x_1^3 - \alpha x_2^2. \tag{3.156}$$

From this we compute the Jacobian matrix at the origin as

$$A(\vec{x}_f = (0, 0)) = \begin{pmatrix} 0 & 1 \\ -2 & \lambda \end{pmatrix}. \tag{3.157}$$

Then the two eigenvalues are computed to

$$\det A_e = e^2 - \lambda e + 2 = 0 \quad \Rightarrow \quad e_\pm = \frac{\lambda}{2} \pm \frac{1}{2}\sqrt{\lambda^2 - 8}. \tag{3.158}$$

The Hopf bifurcation theorem requires that

i) For a particular value of λ the eigenvalues are purely imaginary. Once again this is true for $\lambda = \tilde{\lambda} = 0$ as in that case we find $e_\pm = \pm i\sqrt{2}$.

ii) We compute

$$\left.\frac{d}{d\lambda}\operatorname{Re}(e_\pm(\lambda))\right|_{\lambda=\tilde{\lambda}=0} = \frac{1}{2} > 0. \tag{3.159}$$

iii) We have to show that the origin is asymptotically stable, which we do by means of the corollary. For $\lambda = \tilde{\lambda} = 0$ the corresponding eigenvectors to the eigenvalues $e_{\pm} = \pm i\sqrt{2}$ are $\vec{v}_{\pm} = (\mp i, \sqrt{2})$. Using now these eigenvectors as column vectors in the matrix U, which is used for the similarity transfomation will not yield the form the Jordan form (3.147) with $\omega = \sqrt{2}$. Let us therefore compute U in a pedestrian way. Making a generic ansatz for U, we have to solve

$$U^{-1}AU = J \quad \Leftrightarrow \quad \begin{pmatrix} 0 & 1 \\ -2 & \lambda \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & \sqrt{2} \\ -\sqrt{2} & 0 \end{pmatrix}, \tag{3.160}$$

which gives

$$\begin{pmatrix} c & d \\ -2a & -2b \end{pmatrix} = \begin{pmatrix} -\sqrt{2}b & \sqrt{2}a \\ -\sqrt{2}d & \sqrt{2}c \end{pmatrix}. \tag{3.161}$$

Comparing this equality entry by entry we obtain $c = -\sqrt{2}b$ and $d = \sqrt{2}a$ for $\lambda = 0$. Making a convenient choice $a = 1$ and $b = 0$, we find

$$U = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{2} \end{pmatrix}, \tag{3.162}$$

which means according to $\vec{x} = U\vec{y}$ we have to replace in (3.155), (3.156) $x_1 \to y_1$ and $x_2 \to \sqrt{2}y_2$ in order to obtain the system (3.148). Then the original system (3.155), (3.156) is converted into the form

$$\dot{y}_1 = \sqrt{2}y_1, \tag{3.163}$$

$$\sqrt{2}\dot{y}_2 = -y_1^2 y_2 \sqrt{2} - 2y_1 - y_1^3 - \alpha 2\sqrt{2} y_2^3. \tag{3.164}$$

More conveniently (3.164) is converted into

$$\dot{y}_2 = -y_1^2 y_2 - \sqrt{2}y_1 - y_1^3/\sqrt{2} - 2\alpha y_2^3.$$

From (3.163), (3.164) we compute the stability index. The only non-vanishing term in the definition for I are $Y_{112}^2 = -2$ and $Y_{222}^2 = -12\alpha$. With $\omega = \sqrt{2}$ the stability index is therefore computed to

$$I = \omega \left(Y_{112}^2 + Y_{222}^2 \right) = \sqrt{2}(-2 - 12\alpha). \tag{3.165}$$

Clearly for $\alpha = 1$ this is negative. Therefore it follows by means of the corollary that the origin is asymptotically stable.

Thus the Hopf bifurcation theorem applies, which means in particular that $(0, 0, 0)$ is a bifurcation point for the system (3.155), (3.156). Since for $\lambda > 0$ the origin is surrounded by stable limit cycles and the differential equation (3.154) is oscillatory for some $\lambda > 0$.

When $\alpha = -1$ the stability index is positive and we can not conclude that the origin is asymptotically stable and therefore can not apply the Hopf bifurcation theorem.

3.8 Hamiltonian Systems

Hamiltonian systems are a particular subclass of dynamical systems, which play a central role in physics. When one considers a dynamical system with N *degrees of freedom*, the following differential equations hold when the system can be decribed by a Hamiltonian formalism

$$\frac{d\vec{q}}{dt} = \frac{\partial H(\vec{p},\vec{q})}{\partial \vec{p}} \qquad \text{and} \qquad \frac{d\vec{p}}{dt} = -\frac{\partial H(\vec{p},\vec{q})}{\partial \vec{q}}. \tag{3.166}$$

The function $H(\vec{p},\vec{q})$ is called the *Hamiltonian* of the system, depending on the coordinates $\vec{q} = (q_1, q_2, \ldots, q_n)$ and momenta $\vec{p} = (p_1, p_2, \ldots, p_n)$. The degrees of freedom specify the dimension of the phase space, which means considering the system (3.166) in three dimensional space implies $N = 6n$. Each particle has six degrees of freedom, i.e. three related to its position in three dimensional space and three related to its momentum. For instance, the case $n = 3$ corresponds to the famous three-body problem. The central question in classical mechanics is to describe the system of the above type in phase space. We make here one crucial assumption, namely we exclude friction, which otherwise would lead to a non-trivial loss of energy in our system. Our aim is to analyse these systems in a similar manner as we have analysed the more generic type of dynamical systems, but we want to exploit the special form of these systems. As before, of particular interest is the question of what can be said say about such systems when one does not know the exact solution? Let us restrict once more to the case $n = 1$ in one dimension, that is $N = 2$.

Definition: *A system of differential equations on* $\mathbb{R}^2$ *is said to be a* <u>Hamiltonian system</u> *with two degrees of freedom if there exists a twice continuously differentiable function* $H(x_1, x_2)$ *such that*

$$\dot{x}_1 = \frac{\partial H}{\partial x_2} \qquad \text{and} \qquad \dot{x}_2 = -\frac{\partial H}{\partial x_1}. \tag{3.167}$$

The equations (3.167) are said to be the *equations of motions* corresponding to the Hamiltonian H. When H does not depend explicitly on the time t, i.e. it is of the form $H(x_1(t), x_2(t))$ and not $H(x_1(t), x_2(t), t)$, the system is called *autonomous*.

Example 1: One of the most famous examples is the harmonic oscillator, decribed by the Hamiltonian

$$H(x_1, x_2) = \frac{1}{2}\left(x_2^2 + \omega^2 x_1^2\right) = \left(\frac{1}{2}\left(p^2 + \omega^2 q^2\right)\right) \qquad \omega \in \mathbb{R}. \tag{3.168}$$

Translating to the notions of physics we identify $x_2 \equiv p$ and $x_1 \equiv q$. In classical mechanics this describes the motion of a particle obeying Hook's law, i.e. a particle which is subject to a linear force towards its equilibrium position, as for instance a particle swinging on a spring. Given the Hamiltonian (3.168) we can derive from the defining relations for a Hamiltonian system (3.167) the corresponding dynamical system (equation of motion)

$$\dot{x}_1 = \frac{\partial H}{\partial x_2} = x_2 \qquad \text{and} \qquad \dot{x}_2 = -\frac{\partial H}{\partial x_1} = -\omega^2 x_1. \tag{3.169}$$

The solutions to (3.169) are easily found to be

$$x_1 = A\sin(\omega t + \phi) \qquad \text{and} \qquad x_2 = A\omega\cos(\omega t + \phi) \qquad A, \phi \in \mathbb{R}. \tag{3.170}$$

We may verify this also by combining the two equations in (3.169) into one single equation $\ddot{x}_1 + \omega^2 x_1 = 0$. The constants A, ϕ incorporate the initial conditions.
Example 2: Let us next perturb the harmonic oscillator and consider

$$H(x_1, x_2) = \frac{1}{2}\alpha x_1^2 + \beta x_1 x_2 + \frac{1}{2}\gamma x_2^2 \qquad \alpha, \beta, \gamma \in \mathbb{R}, \tag{3.171}$$

from which we compute the dynamical system

$$\dot{x}_1 = \frac{\partial H}{\partial x_2} = \beta x_1 + \gamma x_2 \qquad \text{and} \qquad \dot{x}_2 = -\frac{\partial H}{\partial x_1} = -\alpha x_1 - \beta x_2. \tag{3.172}$$

We saw that it is easy to compute the equations of motion from a given Hamiltonian, but what about the reverse situation? A dynamical system might not even correspond to a Hamiltonian system since (3.167) is of a very special form. The following proposition is a criterium to decide that.
Proposition 1: *A dynamical system*

$$\dot{x}_1 = F_1(x_1, x_2) \qquad \text{and} \qquad \dot{x}_2 = F_2(x_1, x_2), \tag{3.173}$$

is a Hamiltonian system if and only if

$$\operatorname{div} \vec{F} = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} = 0. \tag{3.174}$$

Proof:

"⇒" Let us suppose the dynamical system is a Hamiltonian system. Then by definition there exists a functions H with $F_1 = \partial H/\partial x_2$ and $F_2 = -\partial H/\partial x_1$. Therefore

$$\operatorname{div} \vec{F} = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} = \frac{\partial^2 H}{\partial x_1 \partial x_2} - \frac{\partial^2 H}{\partial x_2 \partial x_1} = 0. \tag{3.175}$$

Note that the differential operators $\partial/\partial x_1$ and $\partial/\partial x_2$ are commutative, meaning that it is the same whether we differentiate first with repect to x_1 and then with respect to x_2 or vice versa.

"⇐" Let us now suppose that $\operatorname{div} \vec{F} = 0$, then Green's theorem states

$$0 = \iint_{\mathcal{D}} \left[\frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2}\right] dx_1 dx_2 = \oint_{\mathcal{C}} \left[-F_2 dx_1 + F_1 dx_2\right] \tag{3.176}$$

where $\mathcal{D}$ is a simply connected region with boundary $\mathcal{D}$. This means the vectorfield $\vec{F}$ is a gradient field, i.e. there exists a function $H(x_1, x_2)$ with

$$(-F_2, F_1) = \operatorname{grad} H = \left(\frac{\partial H}{\partial x_1}, \frac{\partial H}{\partial x_2}\right), \tag{3.177}$$

which is what we wanted to show □.

Example 1: Consider the dynamical system

$$\dot{x}_1 = x_2 \qquad \text{and} \qquad \dot{x}_2 = x_1. \tag{3.178}$$

We compute for this div $\vec{F} = 0$ and by proposition 1 we deduce that (3.178) is a Hamiltonian system. We can therefore attempt to construct the Hamiltonian

$$\left.\begin{array}{ll} \frac{\partial H}{\partial x_2} = x_2 = \dot{x}_1 & \Rightarrow H(x_1,x_2) = \frac{1}{2}x_2^2 + f(x_1) \\ -\frac{\partial H}{\partial x_1} = x_1 = \dot{x}_2 & \Rightarrow H(x_1,x_2) = -\frac{1}{2}x_1^2 + \tilde{f}(x_2) \end{array}\right\} \Rightarrow H(x_1,x_2) = -\frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + c, \tag{3.179}$$

with $f, \tilde{f}$ being some arbitrary functions and c some constant which we usually set to zero, but it might be fixed by some other value if needed.
Example 2: Consider the dynamical system

$$\dot{x}_1 = x_2 - x_2^2 + x_1^2 \qquad \text{and} \qquad \dot{x}_2 = -x_1 - \alpha x_1 x_2 \quad \alpha \in \mathbb{R}. \tag{3.180}$$

Determine the value for the parameter α such that the system becomes a Hamiltonian system. We compute div $\vec{F} = 2x_1 - \alpha x_1$. According to proposition 1 this means that (3.180) is only a Hamiltonian system when $\alpha = 2$. Note that the Hamiltonian is supposed to be defined on the whole of $\mathbb{R}^2$, such that we do not say here that (3.180) is a Hamiltonian system for $x_1 = 0$.

Having established when a dynamical system is a Hamiltonian system and how to derive the Hamiltonian from it, we will next study some of the properties of these particular types of systems.

3.8.1 Conserved quantities, conservation of energy along a trajectory

In physics Hamiltonians specify the energy of the system. We formulate here everything for $N = 2$, but most statement can be generalised easily to generic values of N.
Theorem: *The autonomous Hamiltonian $H(x_1,x_2)$ is conserved along a trajectory. In other words the total energy $H(x_1,x_2) = E = \text{const}$ (this is how the constant is interpreted in the context of physics) is a first integral and a constant of motion.*
Proof: The derivative of $H(x_1(t), x_2(t))$ along a trajectory $(x_1(t), x_2(t))$ is given by

$$\frac{dH}{dt} = \frac{\partial H}{\partial x_1}\dot{x}_1 + \frac{\partial H}{\partial x_2}\dot{x}_2 = \frac{\partial H}{\partial x_1}\frac{\partial H}{\partial x_2} - \frac{\partial H}{\partial x_2}\frac{\partial H}{\partial x_1} = 0, \tag{3.181}$$

which means $H(x_1,x_2)$ is constant along a solution $(x_1(t), x_2(t))$ and the trajectories lie on contours defined by $H(x_1,x_2) = E = \text{const}$ $\square$.
This means the phase portrait is composed entirely from curves with constant $H(x_1,x_2)$.

We can use this fact to derive a criterium which allows us to decide whether other quantities are preserved in time. For any general function $f(x_1,x_2,t)$ we have

$$\frac{df}{dt} = \frac{dx_1}{dt}\frac{\partial f}{\partial x_1} + \frac{dx_2}{dt}\frac{\partial f}{\partial x_2} + \frac{\partial f}{\partial t} = \frac{\partial H}{\partial x_2}\frac{\partial f}{\partial x_1} - \frac{\partial H}{\partial x_1}\frac{\partial f}{\partial x_2} + \frac{\partial f}{\partial t} =: \{f, H\} + \frac{\partial f}{\partial t}, \tag{3.182}$$

where we have introduced the so-called *Poisson bracket* of f and H. This means the Poisson bracket $\{f, H\}$ characterizes the time-evolution of the function f. To say that the autonomous function $f(x_1, x_2)$ is a constant of motion or conserved in time is therefore equivalent to saying that the Poisson bracket of f and H is vanishing. This is generalized to the definition of the Poisson bracket for any two arbitrary functions $f(x_1, x_2)$ and $g(x_1, x_2)$ to

$$\{f, g\} := \frac{\partial f}{\partial x_1}\frac{\partial g}{\partial x_2} - \frac{\partial f}{\partial x_2}\frac{\partial g}{\partial x_1}. \tag{3.183}$$

It is easy to verify that Poisson brackets satisfy the following properties. Poisson brackets are

i) *linear*

$$\{\kappa f + \lambda g, h\} = \{\kappa f, h\} + \{\lambda g, h\} = \kappa\{f, h\} + \lambda\{g, h\}, \quad \kappa, \lambda \in \mathbb{R}, \tag{3.184}$$

ii) *anti-symmetric*

$$\{f, g\} = -\{g, f\}, \tag{3.185}$$

iii) satisfy the *Jacobi identity*

$$\{\{f, g\}, h\} + \{\{h, f\}, g\} + \{\{g, h\}, f\} = 0, \tag{3.186}$$

iv) and obey the *Leibniz rule*

$$\{f, gh\} = g\{f, h\} + \{f, g\}h. \tag{3.187}$$

We leave it as an exercise to establish these properties. Knowing now how to determine whether a quantity is conserved by means of Poisson brackets, the next theorem provides us with a scheme which allows us to construct new conserved quantities from known ones.

Jacobi-Poisson theorem: *The Poisson bracket of two constants of motion $I_1(x_1, x_2, t)$ and $I_2(x_1, x_2, t)$ is also a constant of motion.*

Proof: According to (3.182) the derivative of the Poisson bracket between $I_1(x_1, x_2, t)$ and $I_2(x_1, x_2, t)$ results to

$$\begin{aligned}
\frac{d}{dt}\{I_1, I_2\} &= \{\{I_1, I_2\}, H\} + \frac{\partial}{\partial t}\{I_1, I_2\}, && (3.188)\\
&= -\{\{H, I_1\}, I_2\} - \{\{I_2, H\}, I_1\} + \left\{\frac{\partial I_1}{\partial t}, I_2\right\} + \left\{I_1, \frac{\partial I_2}{\partial t}\right\}, && (3.189)\\
&= \{\{I_1, H\}, I_2\} - \{\{I_2, H\}, I_1\} + \left\{\frac{\partial I_1}{\partial t}, I_2\right\} - \left\{\frac{\partial I_2}{\partial t}, I_1\right\}, && (3.190)\\
&= \left\{\{I_1, H\} + \frac{\partial I_1}{\partial t}, I_2\right\} - \left\{\{I_2, H\} + \frac{\partial I_2}{\partial t}, I_1\right\}, && (3.191)\\
&= \left\{\frac{dI_1}{dt}, I_2\right\} - \left\{\frac{dI_2}{dt}, I_1\right\}, && (3.192)\\
&= 0. && (3.193)
\end{aligned}$$

From (3.188) to (3.189) we have used the anti-symmetry property (3.185), from (3.189) to (3.190) the Jacobi identity (3.186), from (3.190) to (3.191) the linearity (3.184) and from (3.191) to (3.192) the relation (3.182). The last equality follows from the fact that $I_1(x_1, x_2, t)$ and $I_2(x_1, x_2, t)$ are constants of motion. $\square$

Therefore, given two conserved quantities this theorem can be used to construct additional constants of motion by computing their mutual Poisson bracket. Dynamical systems for which this process can be persued until one has as many constants of motion as degrees of freedom are very special.

Definition: *An autonomous Hamiltonian system with N-degrees of freedom is said to be integrable if it has N independent constants of motion I_j with $1 \leq j \leq N$ which are in involution.*

Definition: *The set of constants of motion I_j is said to be in involution when all their mutual Poisson brackets vanish, i.e.*

$$\{I_i, I_j\} = 0 \qquad \forall i, j \in \{1, 2, \ldots N\}. \tag{3.194}$$

Definition: *The set of constants of motion is said to be independent if none of the I_i can be expressed in terms of the other constants I_j for $i \neq j$.*

The definition for the Poisson brackets (3.183) to a system with higher degree of freedom is easily achieved by

$$\{f, g\} := \sum_{k=1}^{N/2} \frac{\partial f}{\partial q_k} \frac{\partial g}{\partial p_k} - \frac{\partial f}{\partial p_k} \frac{\partial g}{\partial q_k}. \tag{3.195}$$

Let us now return to our standard analysis of the classification of fixed points, limits cycles etc, by making explicitly use of the fact that the systems under consideration are Hamiltonian systems.

3.8.2 Fixed points of Hamiltonian systems

Recall the dynamical system or equation of motions for a Hamiltonian system (3.167). From this follows that the condition for the fixed points read

$$\frac{\partial H}{\partial x_2} = \frac{\partial H}{\partial x_1} = 0, \tag{3.196}$$

which in turn means that the fixed points are stationary points of the Hamiltonian $H(x_1, x_2)$.

Theorem: *Any nondegenerate (that means the Jacobian has nonzero eigenvalues) fixed point of a Hamiltonian system is either a saddle point or a centre.*

Proof: We compute the Jacobian matrix to

$$A = \left. \begin{pmatrix} \frac{\partial^2 H}{\partial x_1 \partial x_2} & \frac{\partial^2 H}{\partial x_2^2} \\ -\frac{\partial^2 H}{\partial x_1^2} & -\frac{\partial^2 H}{\partial x_1 \partial x_2} \end{pmatrix} \right|_{\vec{x}_f} =: \left. \begin{pmatrix} H_{12} & H_{22} \\ -H_{11} & -H_{12} \end{pmatrix} \right|_{\vec{x}_f}. \tag{3.197}$$

The eigenvalues are then obtained from

$$\det(A - \lambda \mathbb{I}) = (H_{12} - \lambda)(-H_{12} - \lambda) + H_{11} H_{22} = 0, \tag{3.198}$$

such that

$$\lambda^2 = -H_{11} H_{22} + H_{12}^2. \tag{3.199}$$

When the fixed point is nondegenerate we only have the two possibilities

$$H_{12}^2 - H_{11}H_{22} \begin{cases} > 0 & \equiv \text{ real eigenvalues of opposite sign } \equiv \text{ saddle point} \\ < 0 & \equiv \text{ purely imaginary eigenvalues } \equiv \text{ centre} \end{cases} , \qquad (3.200)$$

which is what we wanted to prove. □

At the same time, the condition (3.200) provides us with a criterium which allows to determine the nature of a fixed for a Hamiltonian system.

3.8.3 Linear Hamiltonian systems

We have already seen that linear systems are very special as they can be dealt with very systematically, to the extend that their characteristic behaviour is even classifiable. Let us therefore study once more the linear systems which belong to the subclass of Hamiltonian systems.

Example 1: Consider the Hamiltonian

$$H(x_1, x_2) = \frac{1}{2}\kappa^2 x_1^2 + \frac{1}{2}x_2^2 \qquad \kappa \in \mathbb{R}. \qquad (3.201)$$

The corresponding dynamical system is linear

$$\dot{x}_1 = \frac{\partial H}{\partial x_2} = x_2 \qquad \text{and} \qquad \dot{x}_2 = -\frac{\partial H}{\partial x_1} = -\kappa^2 x_1. \qquad (3.202)$$

We compute $H_{11} = \kappa^2$, $H_{22} = 1$ and $H_{12} = 0$. Therefore $H_{12}^2 - H_{11}H_{22} < 0$ and (3.200) yields that the origin is a centre. Since $H(x_1, x_2)$ is conserved along a trajectory $(x_1(t), x_2(t))$ it follows from

$$\frac{1}{2}\kappa^2 x_1^2 + \frac{1}{2}x_2^2 = E, \qquad (3.203)$$

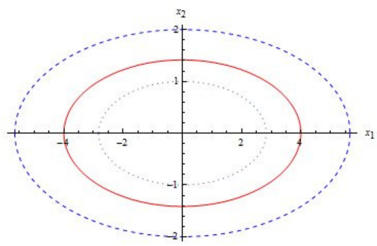

Figure 20: Trajectories for $\kappa = 1/2$ and $E = 0.5, 1, 2$.

that the trajectories are ellipses in the phase plane with axis length $2\sqrt{2E}$ and $2\sqrt{2E}/\kappa$, see figure 20. The direction of time follows from $\dot{x}_1 > 0$ for $x_2 > 0$ and $\dot{x}_1 < 0$ for $x_2 < 0$, i.e. in the upper half plane all trajectories tend to the right and in the lower half plane they all tend to the left. From (3.202) we can also derive a second order differential equation for this system

$$\ddot{x}_1 + \kappa^2 x_1 = 0, \qquad (3.204)$$

which is easily solved.

Example 2: Another standard example for a linear system is the Hamiltonian

$$H(x_1, x_2) = -\frac{1}{2}k^2 x_1^2 + \frac{1}{2}x_2^2 \qquad k \in \mathbb{R}, \qquad (3.205)$$

where we only changed a sign with regard to example 1. The corresponding dynamical system is

$$\dot{x}_1 = \frac{\partial H}{\partial x_2} = x_2 \qquad \text{and} \qquad \dot{x}_2 = -\frac{\partial H}{\partial x_1} = k^2 x_1. \qquad (3.206)$$

Now we compute $H_{11} = -k^2$, $H_{22} = 1$ and $H_{12} = 0$. Therefore $H_{12}^2 - H_{11}H_{22} > 0$ and consequently (3.200) yields that the origin is a saddle point. Since $H(x_1, x_2)$ is conserved along a trajectory $(x_1(t), x_2(t))$ it follows from

$$-\frac{1}{2}k^2 x_1^2 + \frac{1}{2}x_2^2 = E, \tag{3.207}$$

that the trajectories are now hyperbolas in the phase plane, see figure 21. Once again the direction of time follows from $\dot{x}_1 > 0$ for $x_2 > 0$ and $\dot{x}_1 < 0$ for $x_2 < 0$, i.e. in the upper half plane all trajectories tend to the right and in the lower half plane they all tend to the left. From (3.206) we can also derive a second order differential equation for this system

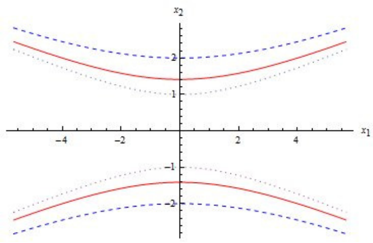

Figure 21: Trajectories for $k = 1/2$ and $E = 0.5, 1, 2$.

$$\ddot{x}_1 - k^2 x_1 = 0 \tag{3.208}$$

which is easily solved.

3.8.4 Potential systems

Let us now be even more restrictive and consider Hamiltonians of a yet more specialised form. The systems to be discussed in this section are ubiquitous in physics and allow for a very intuitive interpretation.

Definition: *A Hamiltonian system which is of the form*

$$H(x_1, x_2) = \frac{1}{2}x_2^2 + V(x_1), \tag{3.209}$$

where $V(x_1)$ is a function which only depends on x_1 and not x_2 is called a potential system with potential (function) $V(x_1)$.

The equations of motion for such a system are

$$\dot{x}_1 = \frac{\partial H}{\partial x_2} = x_2 \qquad \text{and} \qquad \dot{x}_2 = -\frac{\partial H}{\partial x_1} = -\frac{\partial V}{\partial x_1}. \tag{3.210}$$

and therefore the fixed points are determined by

$$x_2 = 0 \qquad \text{and} \qquad \frac{\partial V}{\partial x_1} = 0, \tag{3.211}$$

which means the fixed points are at $(a_i, 0)$ where the a_i are the stationary points of the potential $V(x_1)$. Using now (3.200) we can classify the fixed points with regard to the characteristics of the potential, since now $H_{12}^2 - H_{11}H_{22} = V''(x_1)$. This means

$$V''(x_1) > 0 \quad \Rightarrow \quad \text{centre at } (a,0) \text{ if } V(a) \text{ is a minimum of } V, \tag{3.212}$$

$$V''(x_1) < 0 \quad \Rightarrow \quad \text{saddle point at } (a,0) \text{ if } V(a) \text{ is a maximum of } V. \tag{3.213}$$

We can confirm this result once more from first principles by looking at the linearization. The Jacobian for the system (3.210) is computed to

$$A = \begin{pmatrix} 0 & 1 \\ -V'' & 0 \end{pmatrix}, \tag{3.214}$$

which means the characteristic equation for the eigenvalues is $\lambda^2 + V'' = 0$. This implies that for $V'' < 0$ we have real eigenvalues and therefore a saddle point and for $V'' > 0$ we have purely imaginary eigenvalues and therefore a centre, thus confirming (3.212) and (3.213). We can now collect these informations.

Proposition 2: *The fixed points for the Hamiltonian system described by*

$$H(x_1, x_2) = \frac{1}{2}x_2^2 + V(x_1), \tag{3.215}$$

are located at the points $(a_k, 0)$ *with* $k = 1, 2, 3, \ldots$, *where the* a_k *are stationary points of the potential* $V(x_1)$*. If* $V(a_k)$ *is a minimum then the point* $(a_k, 0)$ *is a centre and if* $V(a_k)$ *is a maximum the point* $(a_k, 0)$ *is a saddle point.*

Let us look at some examples in order to see how to exploit what we found so far.

Example 1: Returning to our examples for linear systems we note that (3.201) was in fact a potential system with $V(x_1) = 1/2\kappa^2 x_1^2$. Clearly at the point $x_1 = 0$ the potential V has a minimum such that from proposition 2 follows that at $(0, 0)$ we have a centre. For (3.205) we have the potential $V(x_1) = -1/2k^2 x_1^2$, for which the point $x_1 = 0$ is a maximum. It follows then from proposition 2 at $(0, 0)$ we have a saddle point, which confirms our previous findings.

Example 2: Let us next consider the potential

$$V(x_1) = \frac{1}{2}x_1^2 - \frac{1}{3}x_1^3. \tag{3.216}$$

We compute for this $V'(x_1) = x_1 - x_1^2$, which means the stationary points obtained from $V'(x_1) = 0$ are $x_1 = 0, 1$. Next we compute from $V''(x_1) = 1 - 2x_1$ that $V''(0) = 1$ and $V''(1) = -1$, which means that there is a minimum of the potential at $x_1 = 0$ and a maximum at $x_1 = 1$. In turn this implies for the phase space that at $(0, 0)$ we have a centre and at $(1, 0)$ we have a saddle point. The corresponding dynamical system is

$$\dot{x}_1 = x_2 \qquad \text{and} \qquad \dot{x}_2 = -\frac{\partial V}{\partial x_1} = x_1^2 - x_1. \tag{3.217}$$

The trajectories are computed by noting from the theorem on energy conservation (3.181) that

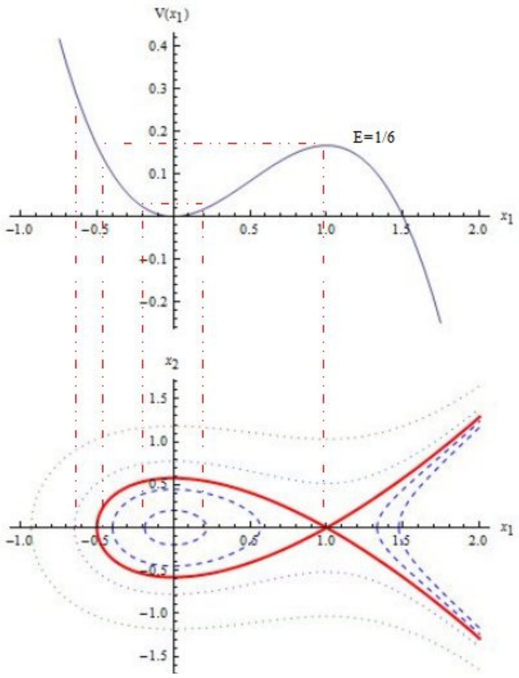

Figure 22: Phase portrait from cubic potential

$$x_2 = \pm\sqrt{2\,[E - V(x_1)]}, \tag{3.218}$$

which follows directly by setting the Hamiltonian in (3.209) equal to E and then solving it for x_2. This means by choosing one particular value for the constant E, we can plot the entire trajectory. The values at $x_2 = 0$ are very distinct because in that case we have

$E = V(x_1)$. (In the notion of physics this means that $p = 0$ and all the kinetic energy is contained in the potential $V(x_1)$.) In order to find the corresponding values for E we have to solve the cubic equation

$$\frac{1}{3}x_1^3 - \frac{1}{2}x_1^2 + E = 0. \tag{3.219}$$

Unlike as for instance for a quadratic equation, there are no simple closed formulae for the real solutions of cubic equations, but there are simple criteria which allow us to make a statement about the amount of real solutions. For this we have to bring the equation into its normal form

$$x_1^3 + rx_1^2 + sx_1 + t = 0. \tag{3.220}$$

The coefficients of this equation are used to define the auxillary quantities

$$p := \frac{3s - r^2}{3} \qquad \text{and} \qquad q := \frac{2r^3}{27} - \frac{rs}{3} + t. \tag{3.221}$$

Using p and q one may then compute the quantity

$$D := \left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2. \tag{3.222}$$

Now one knows that for

$$\begin{array}{l} D > 0 \Rightarrow \exists \text{ one real solution to (3.220)}, \\ D < 0 \Rightarrow \exists \text{ three real solutions to (3.220)}. \end{array} \tag{3.223}$$

We will not derive (3.223) here, but simply employ it[10]. Converting (3.219) into the form (3.220)

$$x_1^3 - \frac{3}{2}x_1^2 + 3E = x_1^3 + rx_1^2 + sx_1 + t = 0 \tag{3.224}$$

we read off $r = -3/2$, $s = 0$ and $t = 3E$. Therefore (3.221) yields $p = -3/4$ and $q = 3E - 1/4$, such that from (3.222)

$$D = \frac{9}{4}E\left(E - \frac{1}{6}\right)\begin{cases} > 0 & \text{for } E < 0 \text{ or } E > 1/6 \\ < 0 & \text{for } 0 < E < 1/6. \end{cases} \tag{3.225}$$

Therefore we have only one real solution when for $E < 0$ or $E > 1/6$ and three real solutions when $0 < E < 1/6$. Notice that depending on the initial condition not all solutions are realized. For instance for $x_1 < 1$ only the two solutions in the valley are energetically possible. In order to reach the third one the particle would have to overcome the potential barrier with $E = 1/6$, which is not possible. Taking the initial condition on the right of the barrier, i.e. $x_1 > 1$, leads just to one realized real soltution as in that case the other two are shield off by the potential barrier.

The separatrix is passing through the saddle point, which we found to be at $(1, 0)$. The equation of the separatrix is therefore determined by

$$H(1,0) = \frac{1}{6} = \frac{1}{2}x_2^2 + \frac{1}{2}x_1^2 - \frac{1}{3}x_1^3, \tag{3.226}$$

[10] For more background and explanations about how this can be derived, see for instance http://en.wikipedia.org/wiki/Cubic_function or http://mathworld.wolfram.com/CubicFormula.html.

such that

$$x_2 = \pm\sqrt{\frac{1}{3} + \frac{2}{3}x_1^3 - x_1^2}. \tag{3.227}$$

The separatrix is indicated by the thick line in figure 25. There are three qualitatively different regimes: Inside the loop of the separatrix all motion is bounded, i.e. there a particle has energy $E < 1/6$ which is less than it needs to overcome the barrier and is therefore trapped by the potential. When it has $E > 1/6$ the motion is unbounded. The particle will just move over the barrier and then fall down the potential and thus becoming faster and faster. When $E < 0$ the motion is also unbounded and the particle will just fall further down the potential. The direction of time follows from $\dot{x}_1 > 0$ for $x_2 > 0$ and $\dot{x}_1 < 0$ for $x_2 < 0$, i.e. in the upper half plane all trajectories tend to the right and in the lower half plane they all tend to the left.

Example 3: Next consider the potential

$$V(x_1) = -\cos x_1. \tag{3.228}$$

We compute for this $V'(x_1) = \sin x_1$, which means the stationary points obtained from $V'(x_1) = 0$ are $x_1^{(n)} = n\pi$ with $n \in \mathbb{Z}$. Next we compute from $V''(x_1) = \cos x_1$ that $V''(2n\pi) = 1$ and $V''[(2n-1)\pi] = -1$, which means that there are minima of the potential at $x_1 = 2n\pi$ and maxima at $x_1 = 1$. By proposition 2 this implies for the phase space that at $(2n\pi, 0)$ we have a centres and at $((2n-1)\pi, 0)$ we have a saddle points. The corresponding dynamical system is

$$\dot{x}_1 = x_2 \qquad \text{and} \qquad \dot{x}_2 = -\frac{\partial V}{\partial x_1} = -\sin x_1, \tag{3.229}$$

such that we obtain the second order differential equation

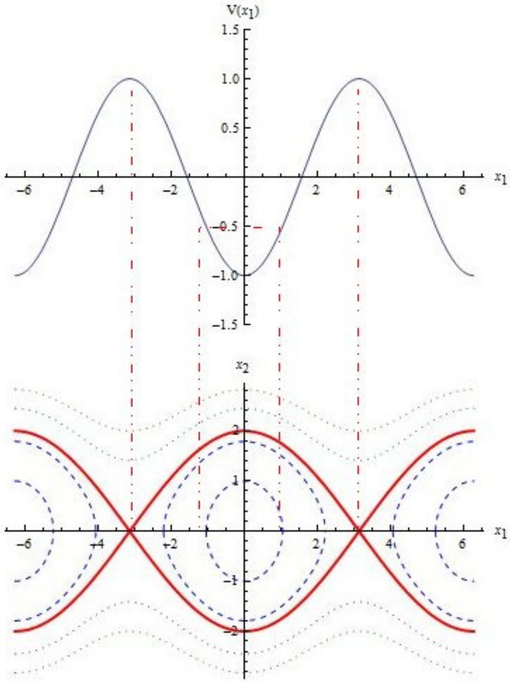

Figure 23: Phase portrait from cosine-potential

$$\ddot{x}_1 + \sin x_1 = 0. \tag{3.230}$$

Since the separatrices pass through the saddle points, the determing equation for them is

$$H[(2n-1)\pi, 0] = -\cos[(2n-1)\pi] = 1 = \frac{1}{2}x_2^2 - \cos x_1. \tag{3.231}$$

Therefore the separatrix is given by

$$x_2^2 = 2(1 + \cos x_1) = 4\cos^2\left(\frac{x_1}{2}\right) \quad \Rightarrow \quad x_2 = \pm 2\cos\left(\frac{x_1}{2}\right). \tag{3.232}$$

We indicated the separatrix in figure 23 by the thick line. When the energy constant is $E < 1$ the motion is inside the separatrix and is bounded. The particle is trapped inside the different values of the potential. Whereas when $E > 1$ the motion is unbounded. The particle will just move over all maxima and just slows down a bit when it passes a valley. As always in potential systems, the direction of time follows from $\dot{x}_1 > 0$ for $x_2 > 0$ and $\dot{x}_1 < 0$ for $x_2 < 0$, i.e. in the upper half plane all trajectories tend to the right and in the lower half plane they all tend to the left.

Example 4: Next consider some generic potential such as the one depicted in figure 24.

It is clear from the above example that we can sketch the phase portraits related to any type of potential, even if we do not know the function explicitly. We can identify the maxima and minima graphically and indicate the the corresponding saddle points and centres, respectively. For energy values below maxima, i.e. inbetween potential barriers, we always have bounded motion. When the energy constant has a value above a certain potential barrier we have an unbounded motion with regard to that barrier, but the motion might be bounded by an additional maxima, as exemplified in figure 24. We also know the phase portrait is symmetric about the x_1-axis. Since we always have $\dot{x}_1 = x_2$ for potential systems, the direction of time of all trajectories in the upper half plane is always to the right and in the lower half plane always to the left. More explicitly we can compute all trajectories from (3.218) for fixed values of the energy constant.

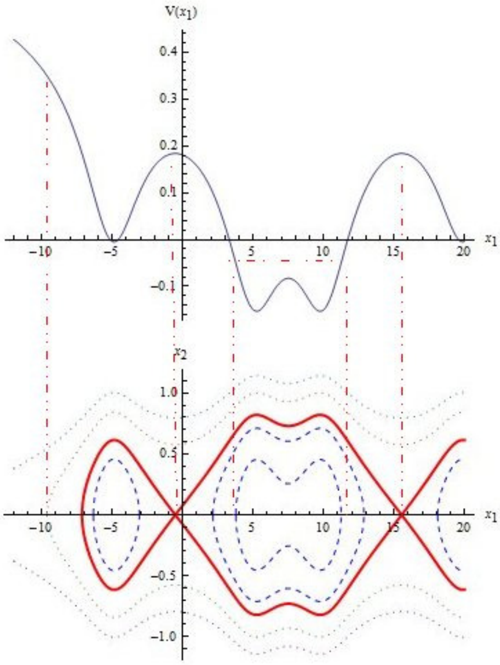

Figure 24: Phase portrait from arbitray potential

Example 5: Show that the equation $\ddot{x} + x + x^3 = 0$ has only periodic solutions.

Using the transformation (3.2), we first convert this equation into two first order equations

$$\dot{x}_1 = \dot{x} = x_2, \tag{3.233}$$

$$\dot{x}_2 = -x_1 - x_1^3. \tag{3.234}$$

The fixed points are computed easily from $x_2 = 0$ and $x_1 + x_1^3 = 0$, i.e. the only fixed point is the origin $(0, 0)$. Let us first check if this system is a Hamiltonian system. According to (3.174) we have to compute

$$\operatorname{div} \vec{F} = \frac{\partial x_2}{\partial x_1} + \frac{\partial}{\partial x_2}(-x_1 - x_1^3) = 0, \tag{3.235}$$

and conclude that the system is Hamiltonian. We can therefore attempt to compute the

Hamiltonian. We find

$$\left.\begin{array}{rlr} \frac{\partial H}{\partial x_2} = x_2 = \dot{x}_1 & \Rightarrow H = \frac{x_2^2}{2} + f(x_1) \\ -\frac{\partial H}{\partial x_1} = -x_1 - x_1^3 = \dot{x}_2 & \Rightarrow H = \frac{x_1^2}{2} + \frac{x_1^4}{4} + \tilde{f}(x_2) \end{array}\right\} \Rightarrow H = \frac{1}{2}x_2^2 + \frac{1}{2}x_1^2 + \frac{1}{4}x_1^4 + c. \quad (3.236)$$

This means the system is also a potential system with potential

$$V(x_1) = x_1^2/2 + x_1^4/4, \quad (3.237)$$

when setting the constant c to zero. We have therefore three possibilities to analyse the system i) treating it as a standard dynamical system, ii) exploiting the fact that it is a Hamiltonian system or iii) making use of the fact that it is also a potential system.

i) Computing the Jacobian matrix at the origin gives

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (3.238)$$

such that the eigenvalues are $\lambda_\pm = \pm i$. The eigenvalues are purely imaginary and therefore the origin is a centre.

ii) Computing $H_{11} = 1 + 3x_1^2\big|_{(0,0)}$, $H_{22} = 1$ and $H_{12} = 0$ gives $H_{12}^2 - H_{11}H_{22} = -1 < 0$, such that according to (3.200) the origin has to be a centre.

iii) The potential $V(x_1) = x_1^2/2 + x_1^4/4$ has a minima a $x_1 = 0$ and therefore we deduce from (3.212) that the origin is a centre. Since the potential tends to inifinity for $x \to \pm\infty$ we conclude that all motion in this potential is bounded and all trajectories are closed. This means all solutions for the Hamiltonian system are periodic and therefore all solutions to the differential equation are periodic.

We note that there is no contradiction between the different viewpoints, but clearly regarding the system as a potential system allows to draw more detailed conclusions. From i) and ii) we deduce properties about the nature of the fixed point, whereas exploiting the fact that the system is also a potential system allows in addition to draw conclusions about the nature of the entire phase portrait.

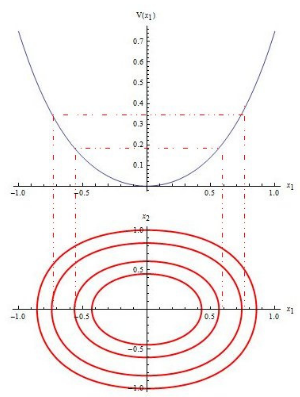

Figure 25: Phase portrait from potential (3.237).

3.8.5 Period of a periodic motion

Let us now have a closer look at example 5 and determine the period of the system, i.e. the time T it takes to move from a particular point (x_1^0, x_2^0) along a trajectory and return to the same point. In order to obtain this time we have to integrate along the closed trajectory, say $\mathcal{C}$, over time

$$T = \oint_{\mathcal{C}} dt = \oint_{\mathcal{C}} \frac{dx_1}{\dot{x}_1} = \oint_{\mathcal{C}} \frac{dx_1}{x_2}. \quad (3.239)$$

Denoting the points where the trajectories intersect the x_1-axis by $(\alpha_1, 0)$ and $(\alpha_2, 0)$ this becomes

$$T = \int_{\alpha_1}^{\alpha_2} \frac{dx_1}{\sqrt{2\left[E - V(x_1)\right]}} + \int_{\alpha_2}^{\alpha_1} \frac{dx_1}{-\sqrt{2\left[E - V(x_1)\right]}} = 2\int_{\alpha_1}^{\alpha_2} \frac{dx_1}{\sqrt{2\left[E - V(x_1)\right]}}, \qquad (3.240)$$

where we have used the relation (3.218). Let us compute this period for the potential in (3.237). First of all we have to determine the intersection points $(\alpha_1, 0)$ and $(\alpha_2, 0)$ with the x_1-axis. For a fixed energy constant E we can find these values from

$$H(\alpha, 0) = E = \frac{1}{2}\alpha^2 + \frac{1}{4}\alpha^4. \qquad (3.241)$$

Since this equation is symmetric in α, i.e. $H(\alpha, 0) = H(-\alpha, 0)$ the period for this particular case becomes

$$T = 2\int_{-\alpha}^{\alpha} \frac{dx_1}{\sqrt{2\left[E - \frac{1}{2}x_1^2 + \frac{1}{4}x_1^4\right]}} = 4\int_{0}^{\alpha} \frac{dx_1}{\sqrt{2\left[E - \frac{1}{2}x_1^2 - \frac{1}{4}x_1^4\right]}}. \qquad (3.242)$$

To be able to evaluate this further we need some initial condition. Taking for instance for corresponding differential equation $x(0) = 1$ and $\dot{x}(0) = 0$ translates into $x_1(0) = 1$ and $x_2(0) = 0$. We can compute the corresponding energy constant for these conditions from

$$H(1, 0) = \frac{1}{2} + \frac{1}{4} = \frac{3}{4} = E, \qquad (3.243)$$

such that (3.242) becomes

$$T = 4\int_{0}^{1} \frac{dt}{\sqrt{2\left[\frac{3}{4} - \frac{1}{2}t^2 + \frac{1}{4}t^4\right]}} = 4\sqrt{2}\int_{0}^{1} \frac{dt}{\sqrt{3 - 2t^2 + t^4}}. \qquad (3.244)$$

This is an elliptic integral of the first kind, which in integral tables is usually given in the form[11]

$$F(\phi, m) = \int_0^{\phi} \frac{d\theta}{\sqrt{1 - m\sin^2\theta}}. \qquad (3.245)$$

Using a variable transformation we can relate this to the form we need in (3.244). Defining $t = \sin\theta$ we have $d\theta = dt/\sqrt{1-t^2}$, such that

$$F(\phi, m) = \int_0^{\arcsin\phi} \frac{dt}{\sqrt{(1-t^2)(1-mt^2)}} = \int_0^{\arcsin\phi} \frac{dt}{\sqrt{1-(1+m)t^2 + mt^4}}. \qquad (3.246)$$

Computing then

$$-iF(i/2\ln 3, -3) = -i\int_0^{i/\sqrt{3}} \frac{dt}{\sqrt{1+2t^2-3t^4}} = \int_0^1 \frac{dt}{\sqrt{3-2t^2+t^4}}, \qquad (3.247)$$

yields therefore the period for our system in form of an elliptic integral of the first kind

$$T = -i4\sqrt{2}F(i/2\ln 3, -3). \qquad (3.248)$$

[11] See for instance http://integrals.wolfram.com/.

3.8.6 Non-potential Hamiltonian systems

Clearly not all Hamiltonian systems are potential systems, but we can still have periodic motions in this case. Let us study the following example for this

$$H(x_1, x_2) = \frac{x_2^2}{2(1+x_1^2)} + \frac{1}{2}x_1^2, \tag{3.249}$$

which is obviously not of the form (3.209) and therefore not a potential system. The equations of motion are derived from (3.167)

$$\dot{x}_1 = \frac{\partial H}{\partial x_2} = \frac{x_2}{(1+x_1^2)} \qquad \text{and} \qquad \dot{x}_2 = -\frac{\partial H}{\partial x_1} = \frac{x_2^2 x_1}{(1+x_1^2)^2} - x_1. \tag{3.250}$$

The fixed point is easily found to be the origin, i.e. $(0,0)$. The nature of the fixed point results from the criterium (3.200). We calculate for this

$$H_{11} = -\frac{\partial}{\partial x_1}\left(\frac{x_2^2 x_1}{(1+x_1^2)^2} - x_1\right) = -\frac{x_2^2}{(1+x_1^2)^2} + \frac{4x_2^2 x_1^2}{(1+x_1^2)^3} + 1, \tag{3.251}$$

$$H_{22} = \frac{\partial}{\partial x_2}\left(\frac{x_2}{(1+x_1^2)}\right) = \frac{1}{(1+x_1^2)}, \tag{3.252}$$

$$H_{12} = -\frac{2x_1 x_2}{(1+x_1^2)^2}, \tag{3.253}$$

Therefore

$$H_{12}^2 - H_{11}H_{22}\big|_{(0,0)} = -1 < 0 \tag{3.254}$$

which implies by (3.200) that the origin is a centre. This means all trajectories are periodic. Let us compute the corresponding period T in (3.239) which is now of course not of the special form (3.240). First we compute the intersections with the x_1-axis from

$$H(\alpha, 0) = E = \frac{1}{2}\alpha^2 \qquad \Rightarrow \alpha = \pm\sqrt{2E}, \tag{3.255}$$

such that we have to demand that these constants are positive, i.e. $E > 0$. The trajectories are found from

$$H(x_1, x_2) = \frac{x_2^2}{2(1+x_1^2)} + \frac{1}{2}x_1^2 = E \qquad \Rightarrow x_2 = \pm\sqrt{(2E - x_1^2)(1+x_1^2)}. \tag{3.256}$$

According to (3.239) the period is therefore

$$T = \oint_{\mathcal{C}} dt = 2\int_{-\alpha}^{\alpha} \frac{dx_1}{\dot{x}_1} = 2\int_{-\sqrt{2E}}^{\sqrt{2E}} \left(\frac{1+x_1^2}{x_2}\right) dx_1 = 2\int_{-\sqrt{2E}}^{\sqrt{2E}} \sqrt{\frac{1+x_1^2}{2E - x_1^2}}\, dx_1 \tag{3.257}$$

$$= 4El(-2E) \qquad \text{for } E > 0, \tag{3.258}$$

where $El(m)$ denotes a complete elliptic intgral of the second kind usually found in integral tables in the form

$$El(m) = \int_0^1 \sqrt{(1 - m\sin^2\theta)}\, d\theta. \tag{3.259}$$