# Dynamical Systems II

### Solutions and marking scheme

INSTRUCTIONS: Full marks correspond to 60 marks.

**1)** Consider the dynamical system

$$\dot{x}_1 = 2\cos x_1 - \cos x_2,$$
$$\dot{x}_2 = 2\cos x_2 - \cos x_1,$$

(*i*) In order to find the fixed points solve $\boxed{2}$

$$2\cos x_1 - \cos x_2 = 0 \qquad \text{and} \qquad 2\cos x_2 - \cos x_1 = 0.$$

We find infinitely many fixed points

$$x_F^{(n,m)} = \left(\pm\frac{\pi}{2} + 2\pi n, \pm\frac{\pi}{2} + 2\pi m\right) \qquad \text{with } n, m \in \mathbb{Z}.$$

(*ii*) In the region $\boxed{6}$

$$\mathcal{D} = \{(x_1, x_2) : -\pi \leq x_1 \leq \pi, -\pi \leq x_2 \leq \pi\} \ ,$$

we have four fixed points

$$x_F^{(1)} = \left(-\frac{\pi}{2}, -\frac{\pi}{2}\right), \quad x_F^{(2)} = \left(-\frac{\pi}{2}, +\frac{\pi}{2}\right), \quad x_F^{(3)} = \left(+\frac{\pi}{2}, -\frac{\pi}{2}\right), \quad x_F^{(4)} = \left(\frac{\pi}{2}, \frac{\pi}{2}\right).$$

We compute the Jacobian matrix to

$$A(x_1, x_2) = \begin{pmatrix} -2\sin x_1 & \sin x_2 \\ \sin x_1 & -2\sin x_2 \end{pmatrix}.$$

Then

$$A(x_F^{(1)}) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \qquad A(x_F^{(2)}) = \begin{pmatrix} 2 & 1 \\ -1 & -2 \end{pmatrix},$$
$$A(x_F^{(3)}) = \begin{pmatrix} -2 & -1 \\ 1 & 2 \end{pmatrix}, \qquad A(x_F^{(4)}) = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$$

Next we compute the eigenvalues of $A(x_F)$

$$\det\left[A(x_F^{(1)})-\lambda\begin{pmatrix}1&0\\0&1\end{pmatrix}\right]=\lambda^2-4\lambda-3=0\Rightarrow\ \lambda_1=3,\lambda_2=1,$$
$$\det\left[A(x_F^{(2)})-\lambda\begin{pmatrix}1&0\\0&1\end{pmatrix}\right]=\lambda^2-3=0\Rightarrow\lambda_{1/2}=\pm\sqrt{3},$$
$$\det\left[A(x_F^{(3)})-\lambda\begin{pmatrix}1&0\\0&1\end{pmatrix}\right]=\lambda^2-3=0\Rightarrow\lambda_{1/2}=\pm\sqrt{3},$$
$$\det\left[A(x_F^{(4)})-\lambda\begin{pmatrix}1&0\\0&1\end{pmatrix}\right]=\lambda^2+4\lambda+3=0\Rightarrow\ \lambda_1=-1,\lambda_2=-3,.$$

This means the linearization theorem can be applied for all fixed points. $x_F^{(1)}$ is an unstable node, $x_F^{(4)}$ is a stable node and $x_F^{(2)}$ and $x_F^{(3)}$ are saddle points.

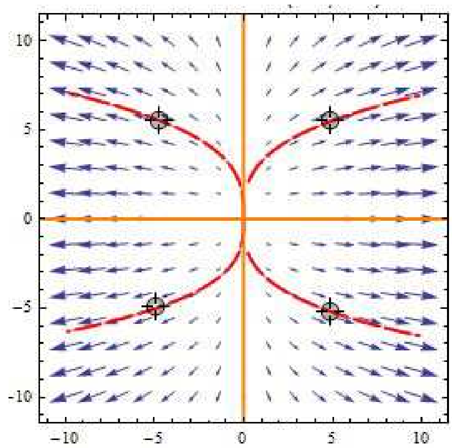
$(iii)$ We compute the eigenvectors for $A(x_F^{(1)})$ to $\boxed{10}$

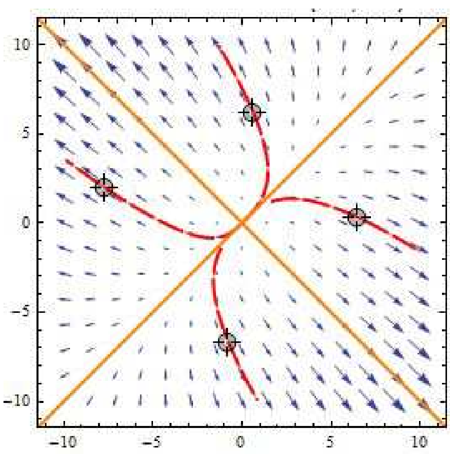
$$v_1^{(1)}=\begin{pmatrix}-1\\1\end{pmatrix}\quad\text{and}\quad v_2^{(1)}=\begin{pmatrix}1\\1\end{pmatrix}.$$

This means the matrix $U^{(1)}=\{v_1^{(1)},v_2^{(1)}\}$ can be used to transform $A$ into the Jordan normal form. Therefore

$$\left(U^{(1)}\right)^{-1}A(x_F^{(1)})U^{(1)}=\begin{pmatrix}3&0\\0&1\end{pmatrix}.$$

The local phase portraits for the linearized system related to $\{\{3,0\},\{0,1\}\}$ and $A(x_F^{(1)})$, respectively, results to:





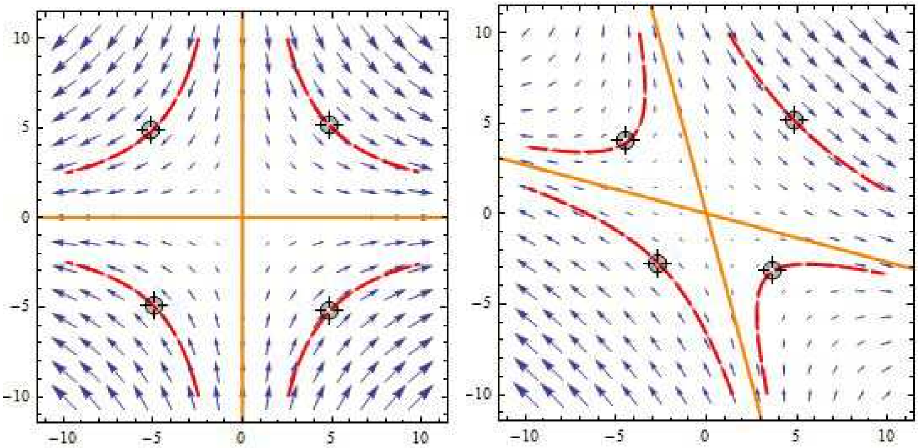

Next we compute the eigenvectors for $A(x_F^{(2)})$ to

$$v_1^{(2)}=\begin{pmatrix}-2-\sqrt{3}\\1\end{pmatrix}\quad\text{and}\quad v_2^{(2)}=\begin{pmatrix}-2+\sqrt{3}\\1\end{pmatrix}.$$

This means the matrix $U^{(2)} = \{v_1^{(2)}, v_2^{(2)}\}$ can be used to transform $A$ into the Jordan normal form. Therefore

$$\left(U^{(2)}\right)^{-1} A(x_F^{(2)}) U^{(2)} = \begin{pmatrix} \sqrt{3} & 0 \\ 0 & -\sqrt{3} \end{pmatrix}.$$

The local phase portraits for the linearized system related to $\{\{\sqrt{3}, 0\}, \{0, -\sqrt{3}\}\}$ and $A(x_F^{(2)})$, respectively, results to:



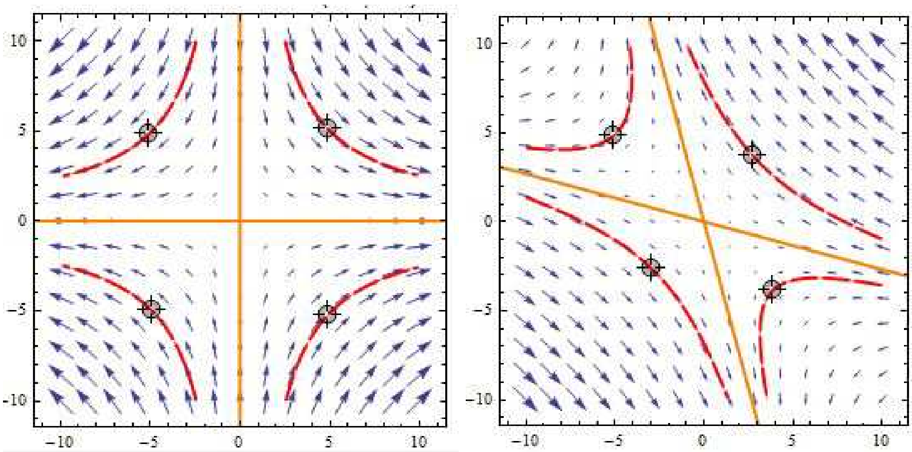
Next we compute the eigenvectors for $A(x_F^{(3)})$ to

$$v_1^{(3)} = \begin{pmatrix} -2+\sqrt{3} \\ 1 \end{pmatrix} \quad \text{and} \quad v_2^{(3)} = \begin{pmatrix} -2-\sqrt{3} \\ 1 \end{pmatrix}.$$

This means the matrix $U^{(3)} = \{v_1^{(3)}, v_2^{(3)}\}$ can be used to transform $A$ into the Jordan normal form. Therefore

$$\left(U^{(3)}\right)^{-1} A(x_F^{(3)}) U^{(2)} = \begin{pmatrix} \sqrt{3} & 0 \\ 0 & -\sqrt{3} \end{pmatrix}.$$

The local phase portraits for the linearized system related to $\{\{\sqrt{3}, 0\}, \{0, -\sqrt{3}\}\}$ and $A(x_F^{(3)})$, respectively, results to:



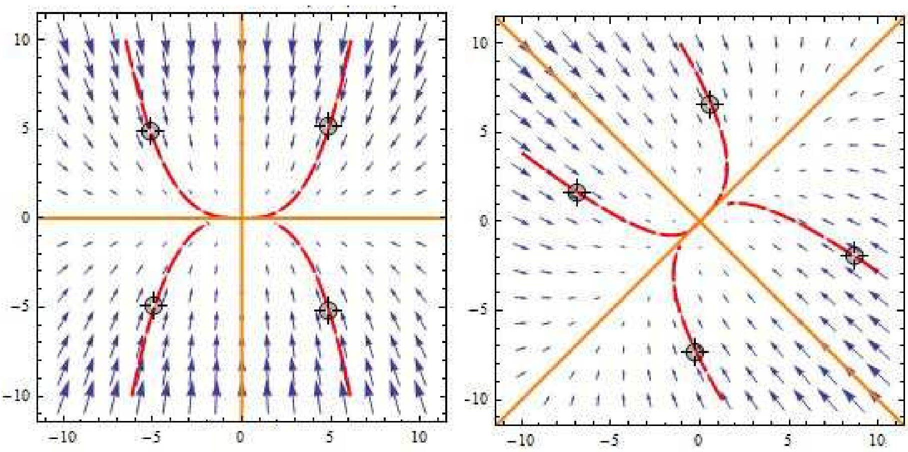
Next we compute the eigenvectors for $A(x_F^{(4)})$ to

$$v_1^{(4)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad v_2^{(4)} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

This means the matrix $U^{(4)} = \{v_1^{(4)}, v_2^{(4)}\}$ can be used to transform $A$ into the Jordan normal form. Therefore

$$\left(U^{(4)}\right)^{-1} A(x_F^{(4)}) U^{(4)} = \begin{pmatrix} -1 & 0 \\ 0 & -3 \end{pmatrix}.$$

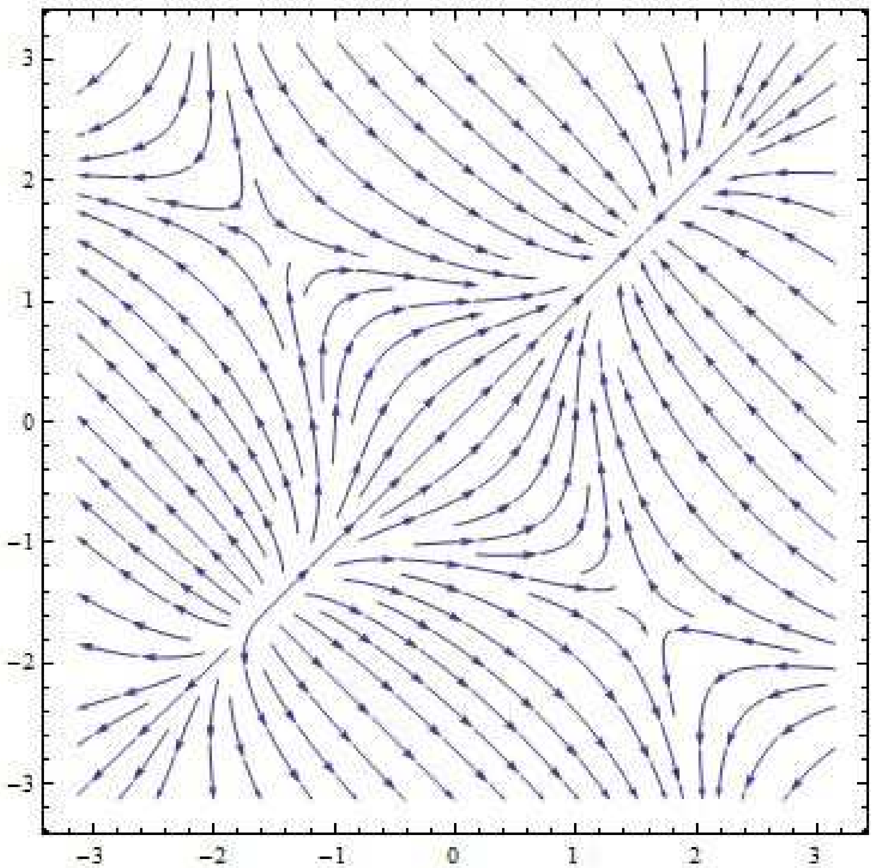
The local phase portraits for the linearized system related to $\{\{-1,0\},\{0,-3\}\}$ and $A(x_F^{(4)})$, respectively, results to:




$(iv)$ The isocline $dx_2/dx_1 = 1$ is computed from $\boxed{1}$

$$\frac{dx_2}{dx_1} = \frac{\dot{x}_2}{\dot{x}_1} = \frac{2\cos x_2 - \cos x_1}{2\cos x_1 - \cos x_2} = 1 \quad \Rightarrow x_1 = x_2.$$

$(v)$ Assemble the information from $(ii)$, $(iii)$, $(iv)$ we obtain $\boxed{6}$




$\boxed{\sum = 25}$

**2)** We have the **Lyapunov stability theorem:** *Consider the system $\dot{\vec{x}} = \vec{F}(\vec{x})$ with a fixed point at the origin. If there exists a real valued function $V(\vec{x})$ in a neighbourhood $N(\vec{x} = 0)$ such that:* $\boxed{5}$

**i)** *the partial derivatives $\partial V/\partial x_1$, $\partial V/\partial x_2$ exist and are continuous,*

**ii)** *the function $V(\vec{x})$ is positive definite,*

**iii)** *$dV/dt$ is negative semi-definite (definite),*

*then the origin is a stable (asymptotically stable) fixed point.*

*A function $V$ for which the conditions i)-iii) hold with iii) (definite) semi-definite is called a (strong) weak Lyapunov function.*

Clearly $i)$ and $ii)$ are satisfied. For the dynamical system

$$\dot{x}_1 = -x_1 + 4x_2, \qquad \dot{x}_2 = -x_1 - x_2^3.$$

and $V(x_1, x_2) = x_1^2 + \lambda x_2^2$ we compute

$$\begin{aligned} \dot{V} = \frac{dV}{dt} &= \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2 \\ &= 2x_1(-x_1 + 4x_2) + 2\lambda x_2\left(-x_1 - x_2^3\right) \\ &= -2x_1^2 + (8 - 2\lambda)x_1 x_2 - 8x_2^4 \end{aligned}$$

$\therefore$ for $\lambda = 4$ we have $\dot{V}(\vec{x} = \vec{0}) = 0$ and $\dot{V}(\vec{x}) < 0$.

$\therefore \exists$ a neighbourhood of the origin in which $\dot{V}$ is negative definite.

$\therefore V[\vec{x}(t)]$ is a strong Lyapunov function on $\mathbb{R}^2$. $\boxed{\sum = 5}$

**3)** We consider the dynamical system

$$\dot{x}_1 = x_1(1 - 4x_1^2 - x_2^2) - \frac{1}{2}x_2(1 + x_1), \tag{1}$$

$$\dot{x}_2 = x_2(1 - 4x_1^2 - x_2^2) + 2x_1(1 + x_1). \tag{2}$$

(*i*) We compute the Jacobian matrix at the origin: $\boxed{4}$

$$A(0,0) = \begin{pmatrix} 1 & -\frac{1}{2} \\ 2 & 1 \end{pmatrix}.$$

The eigenvalues are obtained from

$$\det\left[A(0,0) - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] = \lambda^2 - 2\lambda + 2 = 0 \Rightarrow \ \lambda_\pm = 1 \pm i.$$

This means the origin is an unstable focus.

(*ii*) With $x_1 = r\cos\vartheta$ and $x_2 = r\sin\vartheta$ we obtain $\boxed{15}$

$$\begin{aligned} \dot{x}_1 &= \dot{r}\cos\vartheta - r\sin\vartheta\dot{\vartheta} \\ &= r\cos(\vartheta)\left(-r^2\sin^2(\vartheta) - 4r^2\cos^2(\vartheta) + 1\right) - \frac{1}{2}r\sin(\vartheta)(r\cos(\vartheta) + 1) \qquad (3) \\ \dot{x}_2 &= \dot{r}\sin\vartheta + r\cos\vartheta\dot{\vartheta} \\ &= r\sin(\vartheta)\left(-r^2\sin^2(\vartheta) - 4r^2\cos^2(\vartheta) + 1\right) + 2r\cos(\vartheta)(r\cos(\vartheta) + 1) \qquad (4) \end{aligned}$$

where $x_1^2 + x_2^2 = r^2$. Computing (3) $\times \cos\vartheta$ + (4) $\times$ $\sin\vartheta$ gives

$$\begin{aligned} \dot{r} &= -r^3\sin^4(\vartheta) - 4r^3\cos^4(\vartheta) - 5r^3\sin^2(\vartheta)\cos^2(\vartheta) + \frac{3}{2}r^2\sin(\vartheta)\cos^2(\vartheta) \\ &\quad + r\sin^2(\vartheta) + r\cos^2(\vartheta) + \frac{3}{2}r\sin(\vartheta)\cos(\vartheta) \\ &= -\frac{1}{8}r\left(12r^2\cos(2\vartheta) + 20r^2 - 3r\sin(\vartheta) - 3r\sin(3\vartheta) - 6\sin(2\vartheta) - 8\right) \end{aligned}$$

For $r = 2$ we obtain

$$\dot{r} = \frac{1}{4}(6\sin(\vartheta) + 6\sin(2\vartheta) + 6\sin(3\vartheta) - 48\cos(2\vartheta) - 72)$$

Even if we assume $6\sin(\vartheta) + 6\sin(2\vartheta) + 6\sin(3\vartheta) - 48\cos(2\vartheta) = 66$, which can never happen for the same $\vartheta$, we have $\dot{r} < 0$ for $r = 2$.

For $r = 1/8$ we obtain

$$\dot{r} = \frac{1}{64}\left(\frac{3\sin(\vartheta)}{8} + 6\sin(2\vartheta) + \frac{3}{8}\sin(3\vartheta) - \frac{3}{16}\cos(2\vartheta) + \frac{123}{16}\right)$$

Even if we assume $\frac{3\sin(\vartheta)}{8} + 6\sin(2\vartheta) + \frac{3}{8}\sin(3\vartheta) - \frac{3}{16}\cos(2\vartheta) = -\frac{3}{8} - 6 - \frac{3}{8} - \frac{3}{16} = -\frac{111}{16}$, which can never happen for the same $\vartheta$, we have $\dot{r} > 0$ for $r = 1/8$. Therefore any trajectory which enters the region $\mathcal{D}$ can never leave it. If there is no fixed point in $\mathcal{D}$, then we can employ the Poincaré-Bendixson theorem to deduce that there is at least one limit cycle in $\mathcal{D}$.

We show that $(0,0)$ is the only fixed point. For instance compute RHS of $(2)x_1$−RHS of $(1)x_2 = 0$

$$0 = 2x_1^2(1+x_1) + \frac{1}{2}x_2^2(1+x_1) = \left(2x_1^2 + \frac{1}{2}x_2^2\right)(1+x_1) \Rightarrow x_1 = -1$$

Substituting $x_1 = -1$ into RHS (1)$= 0$ gives $(1 - 4 - x_2^2) = 0$, which does not have a real solution for $x_2$.

(You can also use $x_1 = r/2\cos\vartheta$ and $x_2 = r\sin\vartheta$ which translates into the simpler equation $\dot{r} = r(1-r^2)$.)

*(iii)* For the function $V(x_1,x_2) = \left(1 - 4x_1^2 - x_2^2\right)^2$ and the system (1), (2) we compute $\boxed{7}$

$$\begin{aligned}
\dot{V} &= \frac{dV}{dt} = \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2, \\
&= -16x_1\left(-4x_1^2 - x_2^2 + 1\right)\left(x_1\left(-4x_1^2 - x_2^2 + 1\right) - \frac{1}{2}(x_1+1)\,x_2\right) \\
&\quad -4x_2\left(-4x_1^2 - x_2^2 + 1\right)\left(2x_1(x_1+1) + x_2\left(-4x_1^2 - x_2^2 + 1\right)\right), \\
&= -4\left(4x_1^2 + x_2^2 - 1\right)^2\left(4x_1^2 + x_2^2\right).
\end{aligned}$$

Since $\left(4x_1^2 + x_2^2\right) > 0$ and $\left(4x_1^2 + x_2^2 - 1\right)^2 > 0$ we deduce that $\dot{V} < 0$. This means for $t \to \infty$ we obtain $V(x_1,x_2) = 0$ which means we always reach the ellipse $1 - 4x_1^2 - x_2^2 = 0$. Since no point on the ellipse is a fixed point it must be a limit cycle.

*(iv)* The $\alpha$-limit sets and $\omega$-limit sets are found to be $\boxed{4}$

$$L_\alpha(\vec{x}) = \begin{cases} (0,0) & \text{for } 4x_1^2 + x_2^2 < 1 \\ \varphi_E & \text{for } 4x_1^2 + x_2^2 = 1 \\ \varnothing & \text{for } 4x_1^2 + x_2^2 > 1 \end{cases} \qquad L_\omega(\vec{x}) = \begin{cases} (0,0) & \text{for } r = 0 \\ \varphi_E & \text{for } r \neq 0 \end{cases},$$

where we denoted the ellipse by $\varphi_E$. $\boxed{\sum = 30}$