Dynamical Systems II

Solutions and marking scheme for coursework 2

INSTRUCTIONS: Full marks correspond to 40 marks.

1) (i) *A dynamical system* $\boxed{6}$

$$\dot{x}_1 = F_1(x_1, x_2) \qquad \text{and} \qquad \dot{x}_2 = F_2(x_1, x_2),$$

is a Hamiltonian system if and only if

$$\operatorname{div} \vec{F} = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} = 0.$$

For the given system we therefore have

$$F_1(x_1, x_2) = \alpha\beta x_1^2 x_2^2 \exp[(\alpha+\gamma)x_1^3] + 4x_1 + x_2,$$
$$F_2(x_1, x_2) = \alpha^2 x_1^4 x_2^3 \exp[(\alpha+\gamma)x_1^3] + (a+\beta)x_1 x_2^3 \exp[(\alpha+\gamma)x_1^3] + 2\gamma x_2,$$

We then compute

$$\frac{\partial F_1}{\partial x_1} = 3\alpha\beta x_2^2 x_1^4(\alpha+\gamma)e^{x_1^3(\alpha+\gamma)} + 2\alpha\beta x_2^2 x_1 e^{x_1^3(\alpha+\gamma)} + 4,$$
$$\frac{\partial F_2}{\partial x_2} = 2\gamma + 3\alpha^2 x_2^2 x_1^4 e^{x_1^3(\alpha+\gamma)} + 3x_2^2 x_1(\alpha+\beta)e^{x_1^3(\alpha+\gamma)},$$

such that

$$\frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} = 2(\gamma+2) + x_1 x_2^2 e^{x_1^3(\alpha+\gamma)}\left[3\beta + \alpha\left(2\beta + 3x_1^3(\alpha\beta+\alpha+\beta\gamma)+3\right)\right]$$

This is vanishing for a) $\alpha = 0$, $\beta = 0$, $\gamma = -2$ or b) $\alpha = 9$, $\beta = -9/7$, $\gamma = -2$.

(ii) In order to find the Hamiltonian we now have to integrate $\boxed{4}$

$$\frac{\partial H}{\partial x_2} = F_1(x_1, x_2) \qquad \text{and} \qquad \frac{\partial H}{\partial x_1} = -F_2(x_1, x_2).$$

Case a)

$$\frac{\partial H}{\partial x_2} = 4x_1 + x_2 \qquad \Rightarrow H = 4x_1 x_2 + \frac{x_2^2}{2} + f(x_1)$$
$$\frac{\partial H}{\partial x_1} = 4x_2 \qquad \Rightarrow H = 4x_1 x_2 + \tilde{f}(x_2)$$

Therefore comparing the two expressions

$$H(x_1, x_2) = 4x_1x_2 + \frac{x_2^2}{2} + c.$$

Case b)

$$\begin{aligned}
\frac{\partial H}{\partial x_2} &= -\frac{81}{7}e^{7x_1^3}x_1^2x_2^2 + x_2 + 4x_1 \quad \Rightarrow H = -\frac{27}{7}e^{7x_1^3}x_1^2x_2^3 + \frac{x_2^2}{2} + 4x_1x_2 + f(x_1) \\
\frac{\partial H}{\partial x_1} &= -81e^{7x_1^3}x_2^3x_1^4 - \frac{54}{7}e^{7x_1^3}x_2^3x_1 + 4x_2 \Rightarrow H = -\frac{27}{7}e^{7x_1^3}x_1^2x_2^3 + 4x_1x_2 + \tilde{f}(x_2)
\end{aligned}$$

Therefore comparing the two expressions

$$H(x_1, x_2) = \frac{x_2^2}{2} - \frac{27}{7}e^{7x_1^3}x_1^2x_2^3 + 4x_1x_2 + c.$$

(iii) We integrate $\boxed{4}$

$$\begin{aligned}
\frac{\partial H}{\partial x_2} &= x_2 \quad \Rightarrow H = \frac{x_2^2}{2} + f(x_1) \\
\frac{\partial H}{\partial x_1} &= 2\cosh x_1 \sinh x_1 - 4x_1^3 \Rightarrow H = \frac{1}{2}\cosh(2x_1) - x_1^4 + \tilde{f}(x_2)
\end{aligned}$$

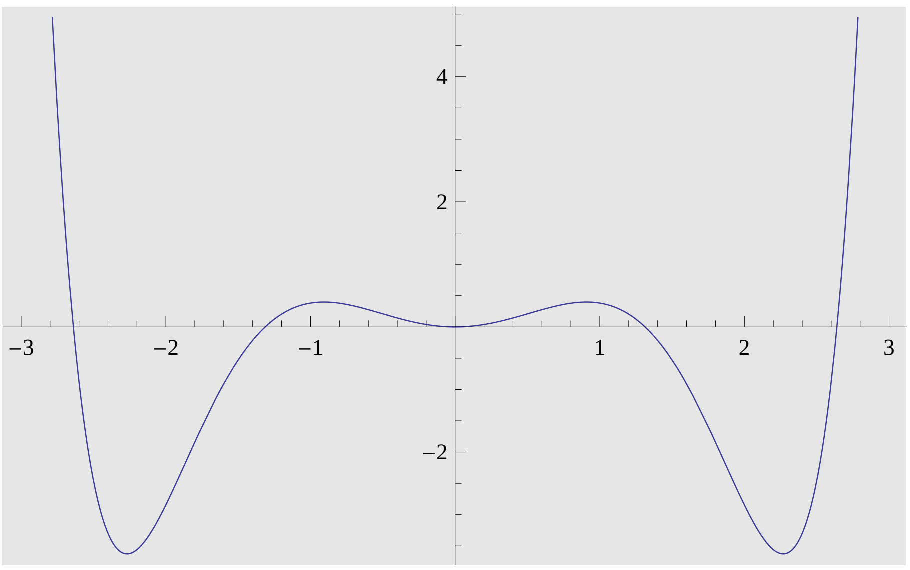
Therefore

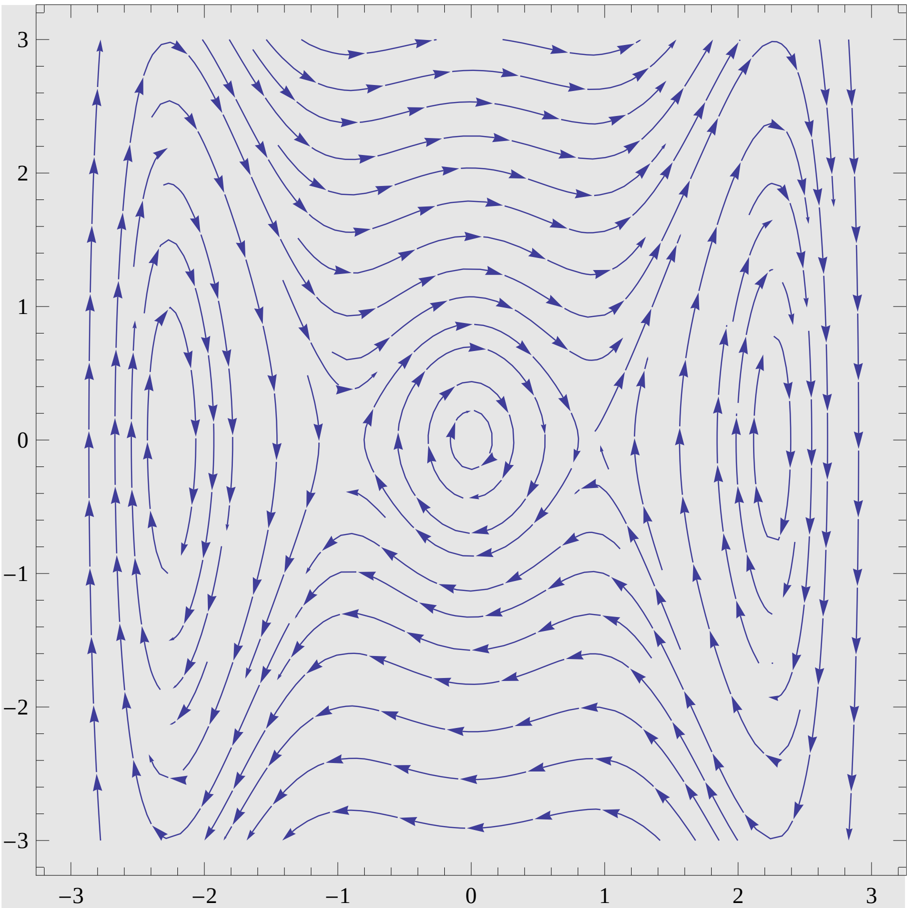
$$H(x_1, x_2) = \frac{x_2^2}{2} + \sinh^2 x_1 - x_1^4 + c$$

This is a potential system with potential $V(x) = \sinh^2 x_1 - x_1^4 + c$.

We require $V(0) = 0$, such that $c = 0$.

(iv) Respectively, the potential and the phase portrait are: $\boxed{6}$





A sketch without the precise values of the minima and maxima is sufficient here. However, a reasoning for the direction of time should be provided, i.e. $\dot{x}_1 > 0$ in the upper half plane and therefore the arrows point to the right, whereas in the lower half plane $\dot{x}_1 < 0$ such that the arrows point to the left. Notice all motion is bounded as the potential tends to infinity for $x \to \pm\infty$. $\boxed{\sum = 20}$

2) (i) The fixed points are found by solving: $\boxed{5}$

$$x = F(x) = 3x - 3\lambda x + \lambda x^2$$

We find $x_1 = 0$ and $x_2 = 3 - 2/\lambda$.
The stability of the fixed point x_f is guaranteed iff $|F'(x_f)| < 1$. With $F'(x) = 3 - 3\lambda + 2\lambda x$ we find

$$\left|F'(x_1)\right| < 1 \qquad \text{for } \frac{2}{3} < \lambda < \frac{4}{3},$$
$$\left|F'(x_2)\right| < 1 \qquad \text{for } 0 < \lambda < \frac{2}{3}.$$

Thus the fixed point $x_2 = 3 - 2/\lambda$ is stable for $0 < \lambda < \frac{2}{3}$ and the fixed point $x_1 = 0$ is stable for $\frac{2}{3} < \lambda < \frac{4}{3}$.

(ii) A 2-cycle is determined by the solution of $\boxed{8}$

$$x = F(F(x)),$$
$$= \lambda^3 x^4 - 6\lambda^3 x^3 + 6\lambda^2 x^3 + 9\lambda^3 x^2 - 21\lambda^2 x^2 + 12\lambda x^2 + 9\lambda^2 x - 18\lambda x + 9x.$$

Since the fixed point also solves this equation we can devide out the factor $F(x) - x$ from $F(F(x)) - x$

$$F(F(x)) - x : F(x) - x = \lambda^2 x^2 + \left(4\lambda - 3\lambda^2\right) x + 4 - 3\lambda.$$

Setting this to zero we find the two points of the two cycle

$$x_\pm = \frac{3}{2} - \frac{2}{\lambda} \pm \frac{\sqrt{3}\sqrt{3\lambda^4 - 4\lambda^3}}{2\lambda^2}$$

For these points to be real we require

$$\lambda \geq \frac{4}{3}.$$

Thus a two cycle exists for $\lambda > \frac{4}{3}$. (Notice for $\lambda = 4/3$ we have $x_+ = x_-$ such that the 2-cycle becomes a one point, i.e. a fixed point.)

(iii) The 2-cycle is stable iff $|G'(x_f)| < 1$ with $G(x) = F(F(x))$. Since $G'(x_+) = G'(x_-) = F'(x_-)F'(x_+)$ we compute $\boxed{7}$

$$F'(x_-)F'(x_+) = 1 + 12\lambda - 6\lambda^2.$$

With

$$\left|1 + 12\lambda - 6\lambda^2\right| < 1 \quad \text{for } \frac{4}{3} < \lambda < \frac{1}{3}(2 + \sqrt{6})$$

we find the domain of stability for the 2-cycle to be $\frac{4}{3} < \lambda < \frac{1}{3}(2 + \sqrt{6})$. $\boxed{\sum = 20}$