Solutions: Dynamical Systems EXAM 2004

1)

$$\ddot{x} + \dot{x} + \alpha x^3 = 0 \qquad \text{for } \alpha \in \mathbb{R}^+$$

i) (2)

$$\begin{aligned} x_1 &= x \\ x_2 &= \dot{x} \end{aligned} \quad \Rightarrow \quad \begin{aligned} \dot{x}_1 &= \dot{x} = x_2 \\ \dot{x}_2 &= \ddot{x} = -x_2 - \alpha x_1^3 \end{aligned}$$

ii) (1) The fixed point is $\vec{x}_f = (0,0)$ $\because$ $\dot{x}_1 = \dot{x}_2 = 0$

(3) Linearisation theorem: Let a nonlinear system have a simple linearisation at some fixed point. Then, in a neighbourhood of the fixed point the phase portraits of the nonlinear system and the one of its linearisation are qualitatively equivalent if the eigenvalues have a real part.

(2) The LT can not be applied as the linearised system is nonsimple, which follows from: Jacobian matrix: $A = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \Rightarrow \det A = 0$

$\Rightarrow$ nonsimple linearis.

iii)

$$V(x_1, x_2) = \beta x_1^4 + \gamma x_2^2 \qquad \beta, \gamma \in \mathbb{R}^+$$

a) the partial derivatives $\frac{\partial V}{\partial x_1}$ and $\frac{\partial V}{\partial x_2}$ exist and are continuous

b) V is positive definite

c) $\dot{V} = \frac{\partial V}{\partial x_1} \dot{x}_1 + \frac{\partial V}{\partial x_2} \dot{x}_2 = -2\gamma x_2^2 + (4\beta - 2\alpha\gamma) x_2 x_1^3$

(3)

$= -2\gamma x_2^2 \qquad$ for $\underline{\beta = \alpha\gamma/2}$

$\leq 0 \quad \Leftrightarrow \quad \dot{V}$ is negative semi-definite in the whole plane

a, b, c $\Rightarrow$ V is a weak Lyapunov fct.

(1) $\Rightarrow$ $\vec{x} = 0$ is a stable fixed point by the

(3) Lyapunov stability theorem: Consider a system $\dot{\vec{x}} = \vec{X}(\vec{x})$ with a fixed point at the origin. If there exists a real valued fct. $V(\vec{x})$ in a neighbourhood $N(\vec{x}=0)$ such that

a) the partial derivatives $\frac{\partial V}{\partial x_1}$ and $\frac{\partial V}{\partial x_2}$ exist and are continuous

b) V is positive definite

c) $\dot V$ is negative semi-definite (definite)

then the origin is a stable (asymptotically stable) fixed point.

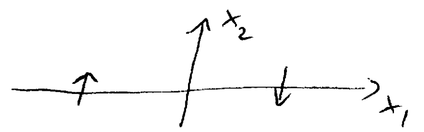
iv) The extension of the LST says:

Let $V(\vec x)$ be a weak Lyapunov fct. for the system $\dot{\vec x} = \vec X(\vec x)$ in

(3) a neighbourhood of an isolated fixed point $\vec x_f = (0,0)$. Then if $\dot V \neq 0$ on a trajectory, except for the fixed point itself, then the origin is asymptotically stable.

- $\dot V(\vec x) = 0$ for $\vec x = (x_1, 0)$

(2) - on this line $\dot x_1 = 0$ $\quad \dot x_2 = -\alpha x_1^3$



$\Rightarrow$ $\vec x = (x_1, 0)$ is not a trajectory

$\Rightarrow$ $\vec x = (0,0)$ is asymptotically stable $\qquad \Sigma$ 20

2)

$$\dot x_1 = x_2 + \alpha\, x_1 \left(1 - \tfrac{\beta}{\alpha} x_1^2 - \tfrac{\beta}{\alpha} x_2^2\right) \qquad \alpha, \beta \in \mathbb{R}^+$$

$$\dot x_2 = -x_1 + \beta\, x_2 \left(1 - x_1^2 - x_2^2\right) \qquad \beta - \alpha < 2$$

i) Jacobian matrix: $A = \begin{pmatrix} \alpha & 1 \\ -1 & \beta \end{pmatrix}$ $\qquad \det A_\lambda = \lambda^2 - (\alpha+\beta)\lambda + \alpha\beta + 1 = 0$

(2) $$\Rightarrow \lambda_\pm = \frac{\alpha+\beta}{2} \pm \frac{1}{2}\sqrt{(\alpha-\beta)^2 - 4}$$

(1) $\Rightarrow \lambda_\pm$ is complex, $\beta, \alpha \in \mathbb{R}^+ \Rightarrow$ unstable focus

ii)

$$\dot r \cos\vartheta - r \sin\vartheta\, \dot\vartheta = r \sin\vartheta + \alpha r \cos\vartheta \left(1 - \tfrac{\beta}{\alpha} r^2\right) \qquad (1)$$

$$\dot r \sin\vartheta + r \cos\vartheta\, \dot\vartheta = -r\cos\vartheta + \beta r \sin\vartheta \left(1 - r^2\right) \qquad (2)$$

(3) (1) $\cos\vartheta$ + (2) $\sin\vartheta$: $\qquad \dot r = r\left(\alpha + \sin^2\vartheta(\beta-\alpha) - \beta r^2\right)$

(2) $\cos\vartheta$ − (1) $\sin\vartheta$: $\qquad r\dot\vartheta = -r + r\beta \sin\vartheta\cos\vartheta(1-r^2) - \alpha r \sin\vartheta\cos\vartheta\left(1 - \tfrac{\beta}{\alpha} r^2\right)$

$$\dot\vartheta = \frac{\beta-\alpha}{2} \sin 2\vartheta - 1$$

(1) $\because \dot\vartheta < 0$ always as $\beta - \alpha < 2 \Rightarrow$ no further fixed point

iii) PBT: Let φ_t be a flow for the system $\dot{\vec{x}} = \vec{X}(\vec{x})$ and let D be a closed, bounded and connected set $D \subset \mathbb{R}^2$ such that $\varphi_t(D) \subset D \;\forall t$. In addition D does not contain any fixed point. Then there exists at least one limit cycle in D.

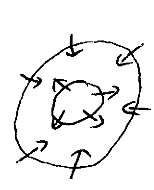
(3)

For $\alpha = 2,\ \beta = 3$: $\quad \dot r = r\left[(2+\sin^2\vartheta) - 3r^2\right] \quad \dot\vartheta = \frac{1}{2}\sin 2\vartheta - 1$

(3)

$r = \frac{1}{\sqrt{3}}$: $\quad \dot r = \frac{1}{\sqrt{3}}\left(1 + \sin^2\vartheta\right) > 0$

$r = 2$: $\quad \dot r = 2\left(2 + \sin^2\vartheta - 12\right) < 0$



$\Rightarrow$ trajectories entering D do not leave it anymore.

(1) $\Rightarrow \exists$ at least one limit cycle in D by PBT

iv)

$\dot r > 0$ means $\quad (2+\sin^2\vartheta) - 3r^2 > 0 \iff r^2 < \frac{2+\sin^2\vartheta}{3}$

on the inner boundary we can make r smaller without spoiling the property

(2) $\Rightarrow r < \sqrt{\min\left(\frac{2+\sin^2\vartheta}{3}\right)} = \sqrt{2/3}$

$\dot r < 0$ means $\quad (2+\sin^2\vartheta) - 3r^2 > 0 \iff r^2 > \frac{2+\sin^2\vartheta}{3}$

on the outer boundary we can make r larger without spoiling

(2) the property $\quad \Rightarrow r > \sqrt{\max\left(\frac{2+\sin^2\vartheta}{3}\right)} = 1$

$\Rightarrow$ define $\quad D^\varepsilon = \{(r,\vartheta) : \frac{\sqrt{2}}{\sqrt{3}} - \varepsilon \le r \le 1 + \varepsilon\} \quad \varepsilon$ small

- we have no fixed point in D^ε

(2)

- $\dot r > 0$ on the inner boundary $\wedge$ $\dot r < 0$ on the outer boundary means trajectories which enter D^ε never leave it

$\because$ $r = \sqrt{2/3}$ and $r = 1$ are not trajectories, the same holds for the regime $\tilde{D} = \{(r, \vartheta) : \sqrt{2/3} \le r \le 1\}$

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3)

$$\dot{x}_1 = (\lambda - 1)x_1 + x_2 + \lambda x_1^2 + 2x_1x_2 + x_1^2 x_2$$

$$\dot{x}_2 = -\lambda x_1 - x_2 - 2x_1x_2 - \lambda x_1^2 - x_1^2 x_2$$

i) linearisation: Jacobian matrix $A^\lambda = \begin{pmatrix} \lambda-1 & 1 \\ -\lambda & -1 \end{pmatrix}$

(1)

(2) $\det A_e^\lambda = (\lambda - 1 - e)(-1-e) + \lambda = 0 \quad \Rightarrow \quad e_\pm = \frac{\lambda-2}{2} \pm \sqrt{\frac{(\lambda-2)^2}{2^2} - 1}$

the eigenvalues are complex for $0 < \lambda < 4$

(1) $\lambda = 2$: eigenvalues are purely complex $\Rightarrow$ centre

(1) The linearisation theorem can not be applied in this case

(1) $0 < \lambda < 2$: eigenvalues complex, negative real part $\Rightarrow$ stable focus

(1) $2 < \lambda < 4$: " " , positive " " $\Rightarrow$ unstable "

(1) $\lambda = 4$: $e_\pm = 1$ $\Rightarrow$ unstable star node

(1) $\lambda = 0$: $e_\pm = -1$ $\Rightarrow$ stable " "

(1) $\lambda > 4$: $e_+ > e_- > 0$ $\Rightarrow$ unstable node

(1) $\lambda < 0$: $e_- < e_+ < 0$ $\Rightarrow$ stable node

ii) $A^{\lambda=2} = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}$ with $U = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$, $U^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$

(1) $\Rightarrow J = U^{-1} A^{\lambda=2} U = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$\vec{x} = U\vec{y} \iff x_1 = y_1 \quad x_2 = -y_1 + y_2$

$\Rightarrow \dot{x}_1 = \dot{y}_1 = y_1 + (y_2 - y_1) + 2y_1(y_2 - y_1) + 2y_1^2 + y_1^2(y_2 - y_1)$

(2)

$\Rightarrow \underline{\dot{y}_1 = y_2 + 2y_1y_2 + y_1^2 y_2 - y_1^3}$

$$\dot{X}_2 = \dot{Y}_2 - \dot{Y}_1 = -2Y_1 - Y_2 + Y_1 - 2Y_1(Y_2 - Y_1) - 2Y_1^2 - Y_1^2(Y_2 - Y_1)$$

$$\Rightarrow \quad \underline{\dot{Y}_2 = -Y_1}$$

To compute the index we need: $w = 1$; $Y_{12}' = 2$; $Y_{111}' = -1$

(2) the remaining derivatives are zero $\Rightarrow I = -1 < 0$

$\Rightarrow$ the origin is asymptotically stable

iii) HBT:
Let $(0, 0, \lambda)$ be a fixed point of the system

(4)

$$\dot{x}_1 = F(x_1, x_2, \lambda)$$
$$\dot{x}_2 = G(x_1, x_2, \lambda) \qquad \lambda \in \mathbb{R}$$

a) The eigenvalues of the linearised system's Jacobian matrix $e_\pm(\lambda)$ are purely imaginary when $\lambda = \tilde{\lambda}$.

b) The real part of the eigenvalues, $Re(e_\pm(\lambda))$, satisfies $\frac{d}{d\lambda} Re\, e_\pm(\lambda)\Big|_{\lambda=\tilde{\lambda}} > 0$

c) The origin is asymptotically stable.

Then
- $\lambda = \tilde{\lambda}$ is a bifurcation point of the system
- for $\lambda \in (\lambda_1, \tilde{\lambda})$ some $\lambda_1 < \tilde{\lambda}$ the origin is a stable fixed point
- for $\lambda \in (\tilde{\lambda}, \lambda_2)$ some $\lambda_2 > \tilde{\lambda}$ the origin is an unstable focus surrounded by a stable limit cycle whose size increases with λ

$\Rightarrow$ This means the system passes a Hopf bifurcation.

a) follows from i)
b) $\frac{d}{d\lambda} e_\pm(\lambda)\Big|_{\lambda=\tilde{\lambda}=2} = \frac{1}{2} > 0$
c) follows from iii)

$\Rightarrow$ the theorem can be applied

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4/ i) A system $\dot{x}_1 = X_1(\vec{x})$, $\dot{x}_2 = X_2(\vec{x})$ is a Hamiltonian system iff

0 $$\operatorname{div} \vec{X} = 0$$

- $\dot{x}_1 = x_2$; $\dot{x}_2 = -x_1^4 + x_1$

1 $\Rightarrow \operatorname{div} \vec{X} = \frac{\partial X_1}{\partial x_1} + \frac{\partial X_2}{\partial x_2} = 0 + 0 \Rightarrow$ Hamiltonian system

- $\dot{x}_1 = x_1^2 + x_2$; $\dot{x}_2 = x_1^3 + 2x_2$

1 $\Rightarrow \operatorname{div} \vec{X} = 2x_1 + 2 \neq 0$ in general $\Rightarrow$ not a HS

ii) $\dot{x}_1 = \frac{\partial H}{\partial x_2} = x_2 \Rightarrow H(x_1, x_2) = \frac{1}{2} x_2^2 + f(x_1)$

2 $\dot{x}_2 = -\frac{\partial H}{\partial x_1} = -x_1^4 + x_1 \Rightarrow H(x_1, x_2) = \frac{1}{5} x_1^5 - \frac{1}{2} x_1^2 + \bar{f}(x_2)$

$$\Rightarrow \underline{H(x_1, x_2) = \frac{1}{2} x_2^2 + \frac{1}{5} x_1^5 - \frac{1}{2} x_1^2}$$

1 $\Rightarrow$ it is a potential system with $V(x_1) = \frac{1}{5} x_1^5 - \frac{1}{2} x_1^2$

- fixed points are at $x_2 = 0$ $\frac{dV}{dx_1} = 0 = x_1^4 - x_1 = x_1(x_1^3 - 1)$

1 $\Rightarrow$ Two fixed point: $x_f^{(1)} = (0,0)$ $x_f^{(2)} = (1,0)$

1 - if $V(x_f)$ is a minimum $\Rightarrow$ the fixed point is a centre

1 if $V(x_f)$ is a maximum $\Rightarrow$ the fixed point is a saddle point

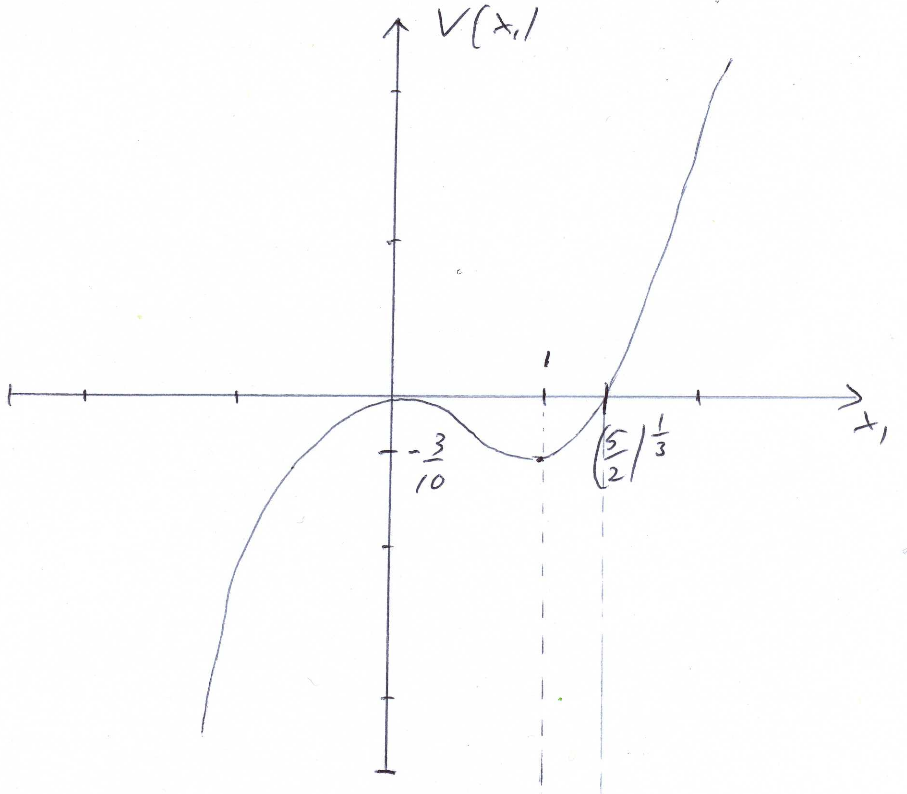
2 $\frac{d^2V}{dx_1^2} = 4x_1^3 - 1 \Rightarrow V''(x_f^{(1)}) = -1 \Rightarrow x_f^{(1)}$ is a maximum of V

$\Rightarrow$ there is a saddle point at $x_f^{(1)}$

2 $\Rightarrow V''(x_f^{(2)}) = 3 \Rightarrow x_f^{(2)}$ is a maximum of V

$\Rightarrow$ there is a centre at $x_f^{(2)}$

(1)



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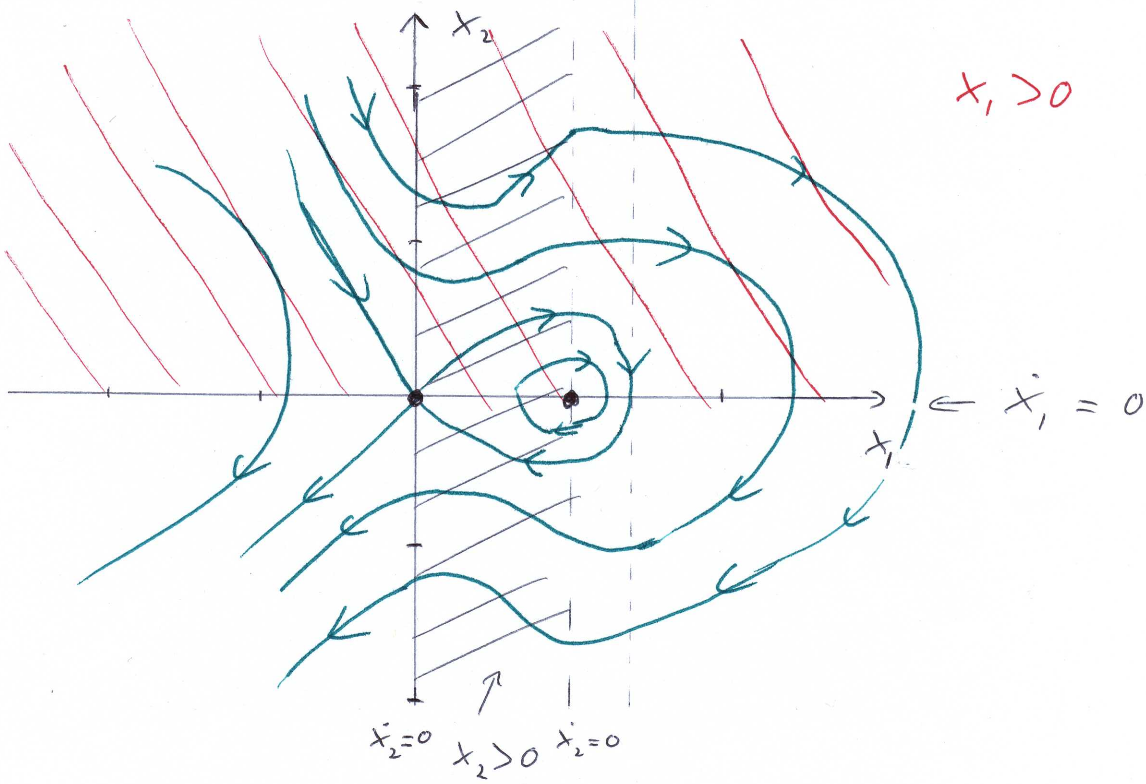

$\dot{x}_1 > 0$ for $x_2 > 0$ \\\

(1)

$\dot{x}_2 > 0$ for $x_1 > 0 \wedge x_1 < 1$ ///

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(2)

- since H is conserved it is constant on a trajectory
- separatrices cross the saddle point $\Rightarrow$ the equation for the seperatrix is $H(0,0) = H(x_1, x_2) = \frac{1}{2}x_2^2 + \frac{1}{5}x_1^5 - \frac{1}{2}x_1^2 \Rightarrow x_2 = \pm x_1\sqrt{1-\frac{2}{5}x_1^3}$

5/

$$x_{n+1} = \lambda(1-\alpha x_n^2) = F(x_n) \qquad \alpha, \lambda \in \mathbb{R}^+$$

i) fixed point: $F(x_f) = x_f \iff x = \lambda - \alpha\lambda x^2$

(2) $\Rightarrow x^2 + \alpha^{-1}\lambda^{-1}x - \alpha^{-1} = 0$

$$\Rightarrow x_\pm = -\frac{1}{2\alpha\lambda} \pm \frac{1}{2\alpha\lambda}\sqrt{1+4\alpha\lambda^2}$$

(1) - x_f is stable if $|F'(x_f)| < 1$

$F'(x) = -2\alpha\lambda x$

(2) $\Rightarrow F'(x_-) = 1 + \sqrt{1+4\alpha\lambda^2} > 1 \Rightarrow$ x_- is always unstable

$F'(x_+) = 1 - \sqrt{1+4\alpha\lambda^2} \Rightarrow 1 - \sqrt{1+4\alpha\lambda^2} > -1$

$1 + 4\alpha\lambda^2 > 4 \Rightarrow \lambda^2 < \frac{3}{4\alpha}$

(2) $\Rightarrow x_+$ is stable for $\lambda < \frac{1}{2}\sqrt{\frac{3}{\alpha}}$

ii) (1) For a 2 cycle we have $F^2(x) = x$

$\Rightarrow x = F(\lambda - \alpha\lambda x^2) = \lambda - \alpha\lambda(\lambda - \alpha\lambda x^2)^2$

(1) $= \lambda - \alpha\lambda^3 - 2\alpha^2\lambda^3x^2 - \alpha^3\lambda^3x^4$

$\iff \underbrace{(\lambda - \alpha\lambda x^2 - x)}_{\text{condition for fixed point}}(1 - \alpha x\lambda - \alpha\lambda^2 + \alpha^2x^2\lambda^2) = 0$

(4)

$\Rightarrow 1 - \alpha x\lambda - \alpha\lambda^2 + \alpha^2x^2\lambda^2 = 0$ is the condition for $F^2(x) = x$

$$\Rightarrow x_\pm^2 = \frac{\alpha\lambda \pm \alpha\lambda\sqrt{4\alpha\lambda^2 - 3}}{2\alpha^2\lambda^2}$$

Now $\alpha = \frac{1}{2}$

$\Rightarrow x_\pm^2 = \frac{1}{\lambda}\left(1 \pm \sqrt{2\lambda^2 - 3}\right)$

(3)

$\Rightarrow$ a 2-cycle exists when $2\lambda^2 - 3 > 0 \Rightarrow \lambda^2 > \frac{3}{2} \Rightarrow \lambda > \sqrt{\frac{3}{2}}$

iii) The stability condition for the 2-cycle is

(1) $$\left| \frac{d}{dx} F^2(x) \Big|_{\hat{x}_\pm} \right| = \left| F'(\hat{x}_+) F'(\hat{x}_-) \right| < 1$$

$F'(x) = -\lambda x$

$$\Rightarrow \left| \lambda^2 \hat{x}_+ \hat{x}_- \right| < 1$$

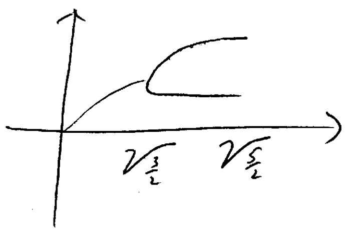
$$\Leftrightarrow \left| 1 - (2\lambda^2 - 3) \right| < 1$$

$$\Leftrightarrow -1 < 4 - 2\lambda^2 < 1$$

$$\Leftrightarrow \frac{3}{2} < \lambda^2 < \frac{5}{2}$$

(2) $$\Leftrightarrow \sqrt{\tfrac{3}{2}} < \lambda < \sqrt{\tfrac{5}{2}}$$

iv)

(1) 

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