1)

$$\dot{x}_1 = -x_1 - x_2^3 \qquad \dot{x}_2 = x_1$$

i) fixed point $\dot{x}_1 = \dot{x}_2 = 0 \Rightarrow x_1 = 0 \Rightarrow -x_2^3 = 0 \Rightarrow x_2 = 0$

[1] $\Rightarrow \vec{x}_f = (0,0)$ is the only fixed point

ii) Linearisation theorem: Let a nonlinear system have a simple linearisation at some fixed point. Then in a neighbourhood of the fixed point the phase portrait of the nonlinear system and the one of its linearisation are qualitatively equivalent if the eigenvalues have a real part. [3]

The LT can not be applied as the linearised system is nonsimple. This follows from:

[2] Jacobian matrix $A(x_1, x_2) = \begin{pmatrix} -1 & -3x_2^2 \\ 1 & 0 \end{pmatrix} \Rightarrow A(\vec{x}_f) = \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}$

$\Rightarrow \det A(\vec{x}_f) = 0 \Rightarrow$ nonsimple linearisation at $\vec{x}_f$.

iii) Let D be a simply connected region of the phase plane in which the function $\vec{X}(\vec{x})$ of the system $\dot{\vec{x}} = \vec{X}(\vec{x})$ has the property that its divergence is of a constant sign, i.e. [3]

$$\operatorname{div} \vec{X} = \frac{\partial X_1}{\partial x_1} + \frac{\partial X_2}{\partial x_2} \quad {>0 \atop <0} \text{ or}$$

Then the system has no closed orbit contained entirely in D.

[2] $\Rightarrow \operatorname{div} \vec{X} = \frac{\partial(-x_1 - x_2^3)}{\partial x_1} + \frac{\partial x_1}{\partial x_2} = -1 < 0$

$\Rightarrow \nexists$ no limit cycle in $\mathbb{R}^2$.

iv) Lyapunov stability theorem: Consider a system $\dot{\vec{x}} = \vec{X}(\vec{x})$

with a fixed point at the origin. If there is a real valued function $V(\vec{x})$ in a neighbourhood $N(\vec{x}=0)$, such that

[3] a) the partial derivatives $\frac{\partial V}{\partial x_1}$ and $\frac{\partial V}{\partial x_2}$ exist and are continuous

b) V is positive definite

c) $\dot{V}$ is negative semi-definite (definite)

then the origin is a stable (asymptotically stable) fixed point.

For $V(x_1, x_2) = 2x_1^2 + 2x_1x_2 + x_2^2 + x_2^4$

[1] a) ✓

[1] b) $V(x_1, x_2) = x_1^2 + (x_1 + x_2)^2 + x_2^4 > 0$ for $\vec{x} \neq (0,0)$

$\Rightarrow$ V is positive definite

c)
$$\dot{V} = \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2$$

$$= (4x_1 + 2x_2)(-x_1 - x_2^3) + (2x_1 + 2x_2 + 4x_2^3)x_1$$

[3]
$$= -4x_1^2 - \underline{4x_1x_2^3} - 2x_1x_2 - 2x_2^4 + 2x_1^2 + 2x_1x_2 + \underline{4x_1x_2^3}$$

$$= -2(x_1^2 + x_2^4) < 0 \quad \text{for } \vec{x}_f \neq (0,0)$$

$\Rightarrow$ $V(x_1, x_2)$ is a strong Lyapunov function

(20)

[1] $\Rightarrow$ The origin is a stable fixed point by the LST.

2)
$$\dot{x}_1 = x_2 + x_1(4 - 5x_1^2 - 5x_2^2)$$
$$\dot{x}_2 = -x_1 + 5x_2(1 - x_1^2 - x_2^2).$$

i) Jacobian matrix at $\vec{x}_f = (0,0)$: $A = \begin{pmatrix} 4 & 1 \\ -1 & 5 \end{pmatrix}$

[2] $\Rightarrow \det A_\lambda = \lambda^2 - 9\lambda + 21 \Rightarrow \lambda_\pm = \frac{9}{2} \pm \frac{1}{2}\sqrt{-3}$

The eigenvalues of A are complex with positive real part.

[1] $\Rightarrow$ The origin is an unstable focus.

ii)

$$\dot r\cos\vartheta - r\sin\vartheta\,\dot\vartheta = r\sin\vartheta + 4r\cos\vartheta\left(1-\tfrac{5}{4}r^2\right) \qquad (1)$$

$$\dot r\sin\vartheta + r\cos\vartheta\,\dot\vartheta = -r\cos\vartheta + 5r\sin\vartheta\left(1-r^2\right) \qquad (2)$$

[3]

$(1)\cos\vartheta + (2)\sin\vartheta$: $\quad \dot r = r\left(4+\sin^2\vartheta - 5r^2\right)$

$(2)\cos\vartheta - (1)\sin\vartheta$: $\quad r\dot\vartheta = -r + 5r\sin\vartheta\cos\vartheta(1-r^2) - 4r\sin\vartheta\cos\vartheta \times\left(1-\tfrac{5}{4}r^2\right)$

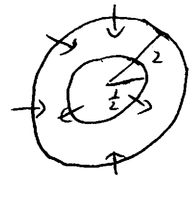
$\Rightarrow \dot\vartheta = \frac{1}{2}\sin 2\vartheta - 1 < 0 \quad \forall r,\vartheta$

$\Rightarrow \exists$ no other fixed point

iii) PBT: Let φ_t be a flow for the system $\dot{\vec x} = \vec X(\vec x)$ and let D be a closed, bounded and connected set $D \subset \mathbb{R}^2$ such that $\varphi_t(D) \subset D \;\forall t$. In addition D does not contain any fixed point. Then there exists at least one limit cycle in D.

[3]

At $r=\frac{1}{2}$: $\quad \dot r = \frac{1}{2}\left(4+\sin^2\vartheta - \frac{5}{4}\right) > 0$



$r = 2$: $\quad \dot r = 2\left(4+\sin^2\vartheta - 20\right) < 0$

[4] $\Rightarrow$ trajectories entering D do not leave it anymore

$\Rightarrow$ by the PBT it follows that there exists at least one limit cycle in D

iv) $\dot r > 0 \iff 4+\sin^2\vartheta - 5r^2 > 0 \iff r^2 < \dfrac{4+\sin^2\vartheta}{5}$

[2] on the inner boundary we can make r smaller without destroying the property $\Rightarrow r < \sqrt{\min\left(\dfrac{4+\sin^2\vartheta}{5}\right)} = \sqrt{\dfrac{4}{5}}$

$\dot{r} < 0 \iff 4 + \sin^2\vartheta - 5r^2 < 0 \iff r^2 > \frac{4+\sin^2\vartheta}{5}$ 4

on the outer boundary we can make r larger

[2] without destroying the property $\Rightarrow r > \sqrt{\max\left(\frac{4+\sin^2\vartheta}{5}\right)} = 1$

$\Rightarrow$ In $D^\varepsilon = \{(r,\vartheta) : \sqrt{\frac{4}{5}} - \varepsilon \le r \le 1+\varepsilon\}$ there is no fixed point

- $\dot{r} > 0$ on inner boundary and $\dot{r} < 0$ on outer boundary, means trajectories entering D^ε do not leave it anymore

[3] $\Rightarrow$ by PBT it follows that there exists at least one limit cycle in D^ε

- $\because$ $r = \sqrt{4/5}$ and $r = 1$ are not trajectories the same conclusion also holds for $\hat{D} = \{(r,\vartheta) : \sqrt{\frac{4}{5}} \le r \le 1\}$

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3) linearisation: $A^\lambda(\vec{x}_f = (0,0)) = \begin{pmatrix} \lambda-1 & 4 \\ -\lambda & -4 \end{pmatrix}$

[1]

$\Rightarrow \det A_e^\lambda = (\lambda-1-e)(-4-e) + 4\lambda = 0$

[2] $\Rightarrow$ $e_\pm = \frac{\lambda-5}{2} \pm \frac{1}{2}\sqrt{\underbrace{\lambda^2-10\lambda+9}_{(\lambda-1)(\lambda-9)}}$

$\Rightarrow$ the eigenvalues are complex for $1 < \lambda < 9$

[1] $\lambda = 5$: eigenvalues are purely complex $\Rightarrow x_f$ is a centre $e_\pm = \pm 2$

[1] The linearisation theorem can not be applied.

[1] $1 < \lambda < 5$: eigenvalues are complex, negative real part $\Rightarrow$ stable focus

$5 < \lambda < 9$: " " " , positive " " $\Rightarrow$ unstable "

[1] $\lambda = 9$; $e_\pm = 2$ $\Rightarrow$ unstable ~~improper~~ node

$\lambda = 1$; $e_\pm = -2$ $\Rightarrow$ stable improper node

[I] $\lambda > 9$: $e_\pm$ real and $e_+ > e_- > 0 \Rightarrow$ unstable node

$\lambda < 1$; $e_\pm$ real and $e_- < e_+ < 0 \Rightarrow$ stable node

ii) $A^{\lambda=5} = \begin{pmatrix} 4 & 4 \\ -5 & -4 \end{pmatrix}$ $\quad U = \begin{pmatrix} 2 & -4 \\ 0 & 5 \end{pmatrix} \Rightarrow x_1 = 2y_1 - 4y_2 \quad x_2 = 5y_2$

Since U is given one knows from i) that

$$J = U^{-1} A U = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} \Rightarrow \underline{\omega = 2}$$

Change from x to y (drop the linear terms as they do not contribute to I:

$$\dot{x}_1 = \frac{5}{2} x_1^2 + 4x_1 x_2 + x_1^2 x_2$$

$$\dot{x}_2 = -\frac{5}{2} x_1^2 - 4x_1 x_2 - x_1^2 x_2$$

[6]

$$\Rightarrow \quad 2\dot{y}_1 - 4\dot{y}_2 = 10y_1^2 + 20y_1^2 y_2 - 40y_2^2 - 80y_1 y_2^2 + 80y_2^3$$

$$5\dot{y}_2 = -10y_1^2 - 20y_1^2 y_2 + 40y_2^2 + 80y_1 y_2^2 - 80y_2^3$$

$$\Rightarrow \quad \underline{\dot{y}_2 = -2y_1^2 - 4y_1^2 y_2 + 8y_2^2 + 16y_1 y_2^2 - 16y_2^3}$$

$$\Rightarrow \quad \dot{y}_1 = 5y_1^2 + 10y_1^2 y_2 - 20y_2^2 - 40y_1 y_2^2 + 40y_2^3$$

$$-8y_1^2 - 16y_1^2 y_2 + 32y_2^2 + 64y_1 y_2^2 - 64y_2^3$$

$$\underline{\dot{y}_1 = -3y_1^2 - 6y_1^2 y_2 + 12y_2^2 + 24y_1 y_2^2 - 24y_2^3}$$

$$\Rightarrow I = 2(2\cdot 24 - 6\cdot 16 - 2\cdot 4) - 6\cdot(-4) + 16(-24) = 2(48 - 96 - 8) + 24 - 16\cdot 24$$

$$= -112 - 15\cdot 24 = -472 < 0 \Rightarrow \text{the origin is asymptotically stable.}$$

iii) HBT: Let $(0,0,\lambda)$ be a fixed point for the system

$$\dot{x}_1 = F(x_1, x_2, \lambda) \qquad \dot{x}_2 = G(x_1, x_2, \lambda) \qquad \lambda \in \mathbb{R}$$

If a) The eigenvalues of the linearised system's Jacobian matrix are purely imaginary when $\lambda = \bar{\lambda}$.

b) The real part of the eigenvalues, $\mathrm{Re}\,(e_\pm(\lambda))$ satisfies

$\frac{d}{d\lambda} \mathrm{Re}\ e_\pm(\lambda)|_{\lambda=\tilde\lambda} > 0$

c) The origin is asymptotically stable

[4] Then
- $\lambda = \tilde\lambda$ is a bifurcation point of the system
- for $\lambda \in (\lambda_1, \tilde\lambda)$ some $\lambda_1 < \tilde\lambda$ the origin is a stable fixed p[oint]
- for $\lambda \in (\tilde\lambda, \lambda_2)$ some $\lambda_2 > \tilde\lambda$ the origin is an unstable fixed point surrounded by a stable cycle whose size increases with λ

$\Rightarrow$ This means the system possesses a Hopf bifurcation.

a) follows from i)

[2] b) $\frac{d}{d\lambda}\ e_\pm(\lambda)|_{\lambda=\tilde\lambda=5} = \frac{1}{2} > 0$ $\Rightarrow$ the theorem applies

c) follows from iii)

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4) i) A system $\dot x_1 = X_1(\vec x)$, $\dot x_2 = X_2(\vec x)$ is a Hamiltonian system iff

[1]
$$\mathrm{div}\, \vec X = 0$$

a)
[1]
$$\mathrm{div}\, \vec X = \frac{\partial x_2}{\partial x_1} + \frac{\partial(-2\cos 2x_1)}{\partial x_2} = 0 \Rightarrow \text{Hamiltonian system}$$

b)
[1]
$$\mathrm{div}\, \vec X = \frac{\partial(2x_1^3 + \cos x_2)}{\partial x_1} + \frac{\partial(3x_1 + \cos x_2)}{\partial x_2}$$

$$= 6x_1^2 - \sin x_2 \neq 0 \quad \text{not a Hamiltonian system}$$

ii)
$$\dot x_1 = \frac{\partial H}{\partial x_2} = x_2 \Rightarrow H(x_1, x_2) = \frac{1}{2}x_2^2 + f(x_1)$$

[2]
$$\dot x_2 = -\frac{\partial H}{\partial x_1} = -2\cos 2x_1 \Rightarrow H(x_1, x_2) = \sin 2x_1 + \tilde f(x_2)$$

$$\Rightarrow \underline{H(x_1, x_2) = \frac{1}{2}x_2^2 + \sin 2x_1 + C} \quad \underline{C = 0}$$

$\Rightarrow$ the system b) is also a potential system, with $V(x_1) = \sin 2x_1$

$\boxed{11}$

(iii) – fixed points:

$$x_2 = 0 \quad \text{and} \quad -2\cos 2x_1 = 0 \qquad \Rightarrow \; 2x_1 = (2n+1)\frac{\pi}{4}$$

$\Rightarrow$ in the range $0 \le x_1 \le 2\pi$ the fixed points are therefore

$\boxed{4}$ $\quad x_f^{(1)} = \left(\frac{\pi}{4}, 0\right) \qquad x_f^{(2)} = \left(\frac{3\pi}{4}, 0\right) \qquad x_f^{(3)} = \left(\frac{5\pi}{4}, 0\right) \qquad x_f^{(4)} = \left(\frac{7\pi}{4}, 0\right)$

– their type is determined by the condition

$$H_{12}^2 - H_{11}H_{22} \begin{cases} > 0 & \equiv \text{saddle point} \\ < 0 & \equiv \text{centre} \end{cases}$$

with $H_{11} = -4\sin 2x_1 \quad ; \quad H_{22} = 1 \quad ; \quad H_{12} = 0$

$\boxed{4}$ $\quad \Rightarrow \; H_{12}^2 - H_{11}H_{22} = 4\sin 2x_1$

$\Rightarrow \; H_{12}^2 - H_{11}H_{22} = 4$ for $x_f^{(1)}, x_f^{(3)}$ $\Rightarrow$ they are saddle points

$\Rightarrow \; H_{12}^2 - H_{11}H_{22} = -4$ for $x_f^{(2)}, x_f^{(4)}$ $\Rightarrow$ they are centres

(iv) The seperatrix crosses the saddle points:

$$H(x_f^{(1)}) = H(x_f^{(3)}) = \sin\frac{\pi}{2} = 1$$

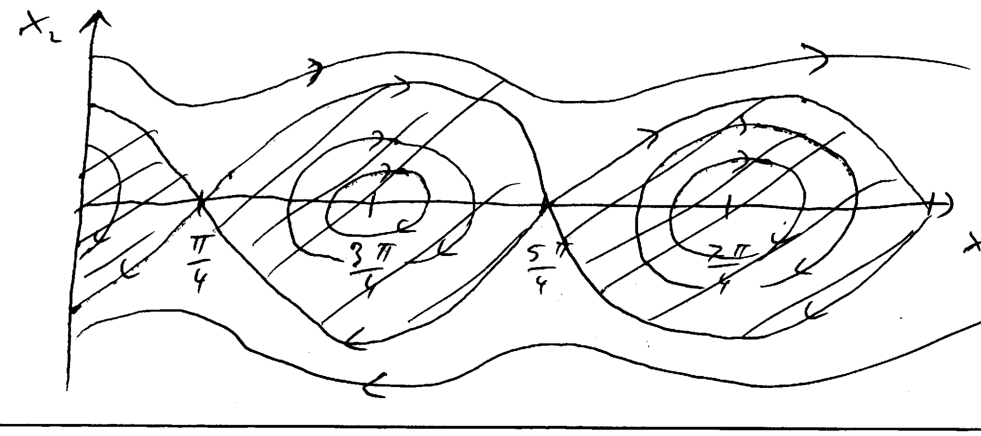
H is conserved along trajectories $\Rightarrow 1 = H(x_1, x_2) = \frac{1}{2}x_2^2 + \sin x_1$

$\boxed{3}$

$\Rightarrow$ the seperatrix is $x_2 = \pm\sqrt{2(1-\sin 2x_1)}$

$\Rightarrow$ phase portrait:

$\boxed{3}$



/// bounded region

– directions from

$x_2 > 0 \Rightarrow \dot{x}_1 > 0$

$x_2 < 0 \Rightarrow \dot{x}_1 < 0$

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5/ i) $$x_{n+1} = F(x_n) = \lambda^2 x_n + (1-\lambda)x_n^2 \qquad \lambda > 1$$

fixed points: $x_n = \lambda^2 x_n + (1-\lambda) x_n^2$

$$\Leftrightarrow \quad (1-\lambda^2)x_n = (1-\lambda)x_n^2$$

$\boxed{2}$
$$\Leftrightarrow \quad (1+\lambda)x_n = x_n^2 \qquad \Rightarrow \qquad \underline{x_n = 0} \quad \text{and} \quad \underline{x_n = 1+\lambda}$$

stability: x_f is a stable fixed point if $|F'(x_f)| < 1$

$\boxed{1}$
$$\Rightarrow \quad |F'(x)| = |\lambda^2 + 2(1-\lambda)x|$$

$$|F'(0)| = |\lambda^2| > 1 \qquad \Rightarrow \quad x_f = 0 \text{ is unstable}$$

$\boxed{3}$
$$|F'(1+\lambda)| = |\lambda^2 + 2(1-\lambda)(1+\lambda)| = |\lambda^2 + 2 - 2\lambda^2| = |2-\lambda^2|$$

stable for $\quad -1 < 2-\lambda^2 < 1$

$$\Leftrightarrow \quad -3 < -\lambda^2 < -1$$

$$\Leftrightarrow \quad 1 < \lambda^2 < 3 \qquad \Rightarrow \quad \underline{1 < \lambda < \sqrt{3}}$$

ii) a two cycle exists when $F(F(x)) = x$

$\boxed{1}$
$$\Rightarrow \quad x = \lambda^2 F(x) + (1-\lambda)(F(x))^2$$

$$\Rightarrow \quad 0 = \lambda^4 x + \lambda^2(1-\lambda)x^2 + (1-\lambda)\left(\lambda^4 x^2 + (1-\lambda)^2 x^4 + 2\lambda^2(1-\lambda)x^3\right) - x$$

this should equal

$$(F(x) - x)\cdot\left(x^2(\lambda-1)^2 + x(1-\lambda)(1+\lambda^2) + (1+\lambda^2)\right) = 0$$

$$\Leftrightarrow \quad x^3\lambda^2(\lambda-1)^2 + x^2\lambda^2(1-\lambda)(1+\lambda^2) + x\lambda^2(1+\lambda^2)$$

$\boxed{4}$
$$+ x^4(1-\lambda)^3 + x^3(1+\lambda^2)(1-\lambda)^2 + \cancel{(1+\lambda^2)}(1-\lambda)x^2$$

$$- x^3(\lambda-1)^2 - x^2\cancel{(1-\lambda)}(1+\lambda^2) - x(1+\lambda^2) \qquad = 0$$

$$\Leftrightarrow \quad x^4(1-\lambda)^3 + 2\lambda^2(1-\lambda)^2x^3 + x^2\lambda^2(1-\lambda)(1+\lambda^2) + x(\lambda^4 - 1) = 0 \ \checkmark$$

solution of condition $\quad x^2 + \dfrac{(1+\lambda^2)}{(1-\lambda)}x + \dfrac{1+\lambda^2}{(1-\lambda)^2} = 0$

$$\Rightarrow \frac{-(1-\lambda)(1+\lambda^2) \pm (\lambda-1)\sqrt{\lambda^4-2\lambda^2-3}}{2(\lambda-1)^2} = x_\pm$$

$\boxed{2}$ For this to be real we require $\lambda^4 > 2\lambda^2+3$

$$\kappa^2 > 2\kappa+3 \Rightarrow \kappa > 3 \Rightarrow \underline{\underline{\lambda > \sqrt{3}}}$$

(iii) the stability condition for a 2-cycle is

$\boxed{1}$ $$\left| \frac{d}{dx} F^2(x)\Big|_{x_\pm} \right| = \left| F'(x_+)\, F'(x_-) \right| < 1$$

$$F'(x) = \lambda^2 + 2(1-\lambda)x$$

$$\Rightarrow F'(x_+)F'(x_-) = \lambda^4 + 2\lambda^2(1-\lambda)\underbrace{(x_-+x_+)}_{\frac{1+\lambda^2}{2(\lambda-1)}} + 4(1-\lambda)^2 \underbrace{x_-x_+}_{\frac{(1+\lambda^2)-(\lambda^4-2\lambda^2-3)}{4(\lambda-1)^2}}$$

$$= \cancel{\lambda^4} - \lambda^2(1+\lambda^2) + (1+\lambda^2)^2 - \cancel{\lambda^4} + 2\lambda^2 + 3$$

$\boxed{6}$ $$= -\lambda^2 - \lambda^4 + 1 + 2\lambda^2 + \lambda^4 + 2\lambda^2 + 3$$

$$= 3\lambda^2 + 4$$

$\Rightarrow$ stable for $|3\lambda^2+4| < 1$

as the existence of a 2-cycle requires $\lambda > \sqrt{3} \Rightarrow |3\lambda^2+4| > 1$

$\Rightarrow$ unstable 2-cycles.

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