Solutions Dynamical Systems EXAM 2006

1)

$$\ddot{x} + \dot{x} + x - \dot{x}x^2 = 0$$

(i) $\left.\begin{array}{l} x_1 = x \\ x_2 = \dot{x} \end{array}\right\} \Rightarrow \begin{array}{l} \dot{x}_1 = \dot{x} = x_2 \\ \dot{x}_2 = \ddot{x} = -x_2 - x_1 + x_2 x_1^2 \end{array}$ (1)

(ii) $(0,0)$ is the only fixed point of the system, since

$\dot{x}_1 = 0 \Rightarrow x_2 = 0$ into $\dot{x}_2$ gives $\dot{x}_2 = -x_1 \Rightarrow x_1 = 0$ (1)

Linearisation theorem: Let a nonlinear system have a simple linearisation at some fixed point. Then in a neighbourhood of the fixed point the phase portrait of the nonlinear system and the one of its linearisation are qualitatively equivalent if the eigenvalues have a real part. (3)

Jacobian matrix: $A(x_1,x_2) = \begin{pmatrix} 0 & 1 \\ -1+2x_1x_2 & -1+x_1^2 \end{pmatrix} \Rightarrow A(0,0) = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$

$\Rightarrow \det A_\lambda = \begin{vmatrix} -\lambda & 1 \\ -1 & -1-\lambda \end{vmatrix} = \lambda^2 + \lambda + 1 = 0 \Rightarrow \underline{\lambda = -\frac{1}{2} \pm \frac{i}{2}\sqrt{3}}$

$\Rightarrow$ The LT can be applied. (3)

$\Rightarrow$ The fixed point is a stable focus.

(iii) Bendixson's criterium: Let D be a simply connected region of the phase plane in which the function $\vec{X}(\vec{x})$ of the system $\dot{\vec{x}} = \vec{X}(\vec{x})$ has the property that its divergence is of a constant sign, i.e.

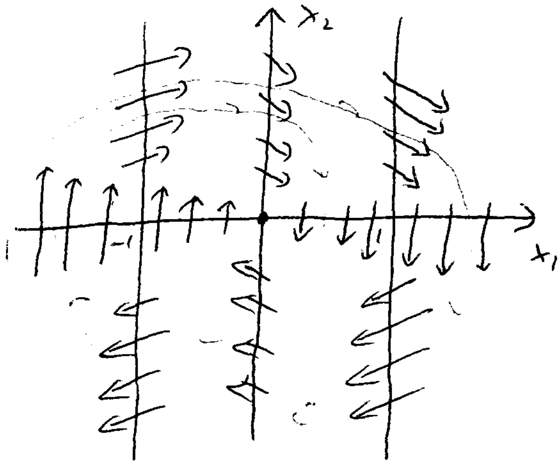
$$\operatorname{div} \vec{X} = \frac{\partial X_1}{\partial x_1} + \frac{\partial X_2}{\partial x_2} \begin{array}{l} > 0 \\ < 0 \end{array} \text{or}, \quad (3)$$

then the system has no closed orbit contained entirely in D.

Theorem: A limit cycle contains at least one fixed point. (1)

For the above system: $\operatorname{div} \vec{X} = x_1^2 - 1$

$\Rightarrow$ There is no limit cycle for $|x_1| < 1$ or $|x_1| > 1$.



$$\left.\begin{aligned}\dot{x}_1 &= 0\\ \dot{x}_2 &= -x_1\end{aligned}\right\} \text{for } x_2 = 0 \qquad \left.\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -x_2\end{aligned}\right\} \text{for } x_1 = 0$$

(6)

$$\left.\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -1\end{aligned}\right\} \text{for } x_1 = 1 \qquad \left.\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= 1\end{aligned}\right\} \text{for } x_1 = -1$$

(iv) Lyapunov stability theorem: Consider a system $\dot{\vec{x}} = \vec{X}(\vec{x})$ with a fixed point at the origin. If there is a real valued function $V(\vec{x})$ in a neighbourhood $N(\vec{x}=0)$ such that

a) the partial derivatives $\partial V/\partial x_1$ and $\partial V/\partial x_2$ exist and are continuous

b) V is positive definite

c) $\dot{V}$ is negative semi-definite (definite) (3)

then the origin is a stable (asymptotically stable) fixed point

For $V(x_1, x_2) = x_1^2 + x_2^2$

a) ✓

b) $V(0,0) = 0$ and $V(x_1, x_2) > 0$ for $\vec{x} \neq (0,0) \Rightarrow V$ is positive definite

c) $\dot{V} = 2x_1\dot{x}_1 + 2x_2\dot{x}_2 = 2x_1x_2 + 2x_2x_1^2 - 2x_1x_2 - 2x_2^2$

$= -2x_2^2(1 - x_1^2)$ (4)

$\Rightarrow \dot{V}$ is negative semi-definite for $|x_1| < 1$

$\therefore$ V is a weak Lyapunov function for $|x_1|<1$

$\therefore$ by the LST: the origin is a stable fixed point

The extension of the LST states: (5)

Let V be a weak Lyapunov function for the system $\dot{\vec{x}} = \vec{X}(\vec{x})$ in a neighbourhood of an isolated fixed point $\vec{x}_f = (0,0)$.

Then if $\dot{V} \neq 0$ on a trajectory, except for the fixed point itself, then the origin is asymptotically stable.

Here $\dot{V}(\vec{x}) = 0$ for $\vec{x} = (x_1, 0)$, but on that line $\dot{x}_1 = 0$, $\dot{x}_2 = -x_1$

$\Rightarrow$ $\vec{x} = (x_1, 0)$ is not a trajectory

$\Rightarrow$ $\vec{x}^0 = (0,0)$ is asymptotically stable in the domain $x_1^2 + x_2^2 < 1$

$\boxed{\sum = 30}$

2) i)

$$\dot{x}_1 = -x_2 + 6x_1(1 - x_1^2 - x_2^2)$$

$$\dot{x}_2 = x_1 + x_2(5 - 6x_1^2 - 6x_2^2)$$

Jacobian matrix at $\vec{x}_f = (0,0)$: $A = \begin{pmatrix} 6 & -1 \\ 1 & 5 \end{pmatrix}$

eigenvalues: $\det(A - \lambda I) = \lambda^2 - 11\lambda + 31 = 0 \Rightarrow \lambda_{1/2} = \frac{11}{2} \pm \frac{i}{2}\sqrt{3}$

$\Rightarrow$ The eigenvalues of A are complex with positive real part.

$\Rightarrow$ The origin is an unstable focus. (5)

ii)

$$\dot{r}\cos\vartheta - r\sin\vartheta\,\dot{\vartheta} = -r\sin\vartheta + 6r\cos\vartheta(1-r^2) \quad (1)$$

$$\dot{r}\sin\vartheta + r\cos\vartheta\,\dot{\vartheta} = r\cos\vartheta + 5r\sin\vartheta(1 - \tfrac{6}{5}r^2) \quad (2)$$

$(1)\times\cos\vartheta + (2)\times\sin\vartheta$: $\dot{r} = 6r\cos^2\vartheta(1-r^2) + 5r\sin^2\vartheta(1-r^2)$

$$\underline{\dot{r}} = 5r + r\cos^2\vartheta - 6r^3 = \underline{r(5 + \cos^2\vartheta - 6r^2)}$$

$(2)\times\cos\vartheta - (1)\times\sin\vartheta$: $r\dot{\vartheta} = r - r\sin\vartheta\cos\vartheta$

$r \neq 0$

$$\Rightarrow \underline{\dot{\vartheta} = 1 - \tfrac{1}{2}\sin 2\vartheta}$$

(6)

Since $\dot{\vartheta} > 0$ is never vanishing then $\exists$ no further fixed points

iii) PBT: Let φ_t be a flow for the system $\dot{\vec{x}} = \vec{X}(\vec{x})$ and let D be a closed, bounded and connected set $D \subset \mathbb{R}^2$ such that $\varphi_t(D) \subset D \; \forall t$. In addition D does not contain any fixed point, then there exist at least one limit cycle in D. (3)

At $r = \frac{1}{2}$: $\quad \dot{r} = \frac{1}{2}\left(5 + \cos^2\vartheta - \frac{6}{4}\right) > 0$

$r = 3$: $\quad \dot{r} = 3\left(5 + \cos^2\vartheta - 6 \cdot 9\right) < 0$

$\Rightarrow$ trajectories entering $D = \{(r,\vartheta) : \frac{1}{2} \leq r \leq 3\}$ do not leave it anymore

$\Rightarrow$ by the PBT it follows that there exists at least one limit cycle in D (4)

iv) $\dot{r} > 0 \iff 5 + \cos^2\vartheta - 6r^2 > 0 \iff r^2 < \dfrac{5+\cos^2\vartheta}{6}$

at the inner boundary we can make r smaller without destroying the property $\Rightarrow r < \sqrt{\min \frac{5+\cos^2\vartheta}{6}} = \sqrt{\frac{5}{6}}$

$\dot{r} < 0 \iff 5 + \cos^2\vartheta - 6r^2 < 0 \iff r^2 > \dfrac{5+\cos^2\vartheta}{6}$

at the outer boundary we can enlarge r without destroying the estimate $\Rightarrow r > \sqrt{\max\left(\frac{5+\cos^2\vartheta}{6}\right)} = 1$ (4)

$\Rightarrow$ In $D^\varepsilon = \{(r,\vartheta) : \sqrt{\frac{5}{6}} - \varepsilon \leq r \leq 1 + \varepsilon\}$ with $0 < \varepsilon \ll 1$ there is no fixed point. Furthermore $\dot{r} > 0$ on the inner boundary and $\dot{r} < 0$ on the outer boundary, which means once again that trajectories entering D^ε do not leave it anymore (4)

$\Rightarrow$ by the PBT it follows that there exists at least one limit cycle in D^ε

- since $r = \sqrt{5/6}$ and $r = 1$ are not trajectories of the system the same conclusion hold for

$D = \{(r,\vartheta) : \sqrt{\frac{5}{6}} \leq r \leq 1\}$. $\boxed{\Sigma : 30}$ (4)

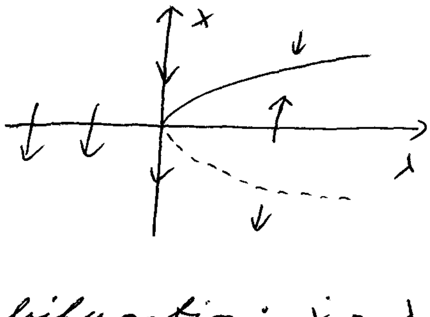
3) i) Bifurcation theory investigates how the number of steady solutions of the system $\dot{\vec{x}} = \vec{X}(\vec{x}, \lambda)$ depend on the parameter λ. A bifurcation occurs if the solutions to $\dot{\vec{x}} = \vec{X}$ change their qualitative behaviour as λ varies. (2)

Def.: Let (x_0, λ_0) be a fixed point for the system $\dot{x} = F(x, \lambda)$. If $\partial F/\partial\lambda|_{(x_0,\lambda_0)} \neq 0$ and $\partial\lambda/\partial x$ changes sign at (x_0, λ_0), then (x_0, λ_0) is called a <u>turning point</u>. (2)

Def.: Let (x_0, λ_0) be a fixed point for the system $\dot{x} = F(x, \lambda)$. If $\partial F/\partial\lambda|_{(x_0,\lambda_0)} = 0$ and $\partial F/\partial x|_{(x_0,\lambda_0)} = 0$ and if through (x_0, λ_0) pass two and only two branches of the equilibrium curve which have both distinct tangents at (x_0, λ_0), then (x_0, λ_0) is called a <u>transcritical bifurcation</u>. (2)

Def.: Let (x_0, λ_0) be a fixed point for the system $\dot{x} = F(x, \lambda)$. If $\partial F/\partial\lambda|_{(x_0,\lambda_0)} = 0$ and $\partial F/\partial x|_{(x_0,\lambda_0)} = 0$ and $\partial\lambda/\partial x$ changes sign on one branch of the equilibrium curve with distinct tangents, then (x_0, λ_0) is called a <u>pitchfork bifurc</u> (2)

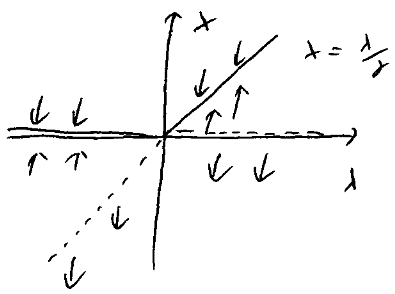
ii) Turning point: $\dot{x} = \lambda - x^2$, $\frac{\partial F}{\partial\lambda} = 1$, $\frac{\partial\lambda}{\partial x} = 2x$ changes sign at x



(4)

transcritical bifurcation: $\dot{x} = \lambda x - \gamma x^2$, $\gamma > 0$

fixed points: $x = 0$ and $x = \lambda/\gamma$



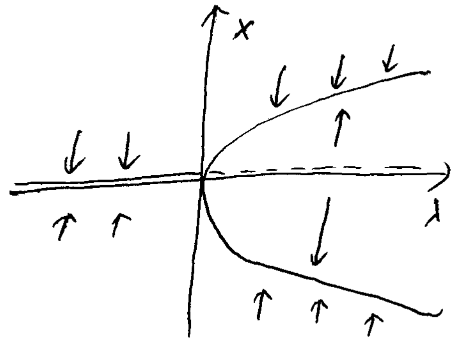
at $(0,0)$: $\left.\frac{\partial F}{\partial x}\right|_{(0,0)} = \lambda - 2\gamma x\Big|_{(0,0)} = 0$

(4)

$\left.\frac{\partial F}{\partial\lambda}\right|_{(0,0)} = x\Big|_{(0,0)} = 0$

pitchfork bifurcation: $\dot{x} = \lambda x - \gamma x^3 \qquad \gamma > 0$

fixed points: $x = 0$ and $x = \pm\sqrt{\lambda/\gamma}$



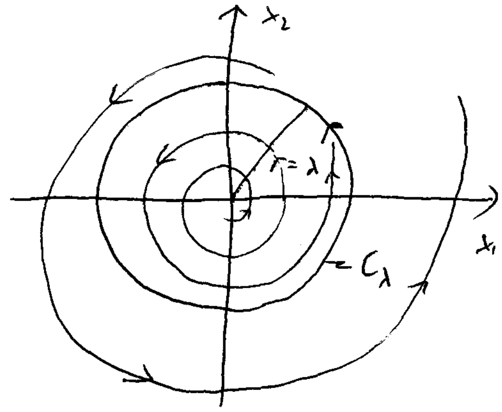
at $(0,0)$: $\left.\frac{\partial F}{\partial x}\right|_{(0,0)} = \lambda - 3\gamma x^2\Big|_{(0,0)} = 0$

$\left.\frac{\partial F}{\partial \lambda}\right|_{(0,0)} = x\Big|_{(0,0)} = 0$ (4)

$\frac{\partial \lambda}{\partial x} = 2\gamma x$ changes sign on $x = \pm\sqrt{\lambda/\gamma}$

(iii) A bifurcation at the origin which changes its characteristic behaviour from stable, to centre, to unstable as the bifurcation parameter varies is called a Hopf bifurcation (1)

(iv)

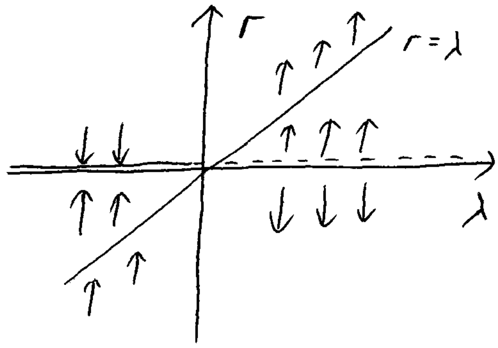


$\dot{r} = 0$ for $r = \lambda$

$\dot{r} > 0$ for $r \neq \lambda$

$$L_\alpha(x) = \begin{cases} 0 & \text{for } 0 \le r < \lambda \\ C_\lambda & \text{for } r \ge \lambda \end{cases}$$

$$L_\omega(x) = \begin{cases} 0 & \text{for } r = 0 \\ C_\lambda & \text{for } 0 < r \le \lambda \\ \phi & \text{for } r > \lambda \end{cases}$$ (4)



$\dot{r} > 0$ for $r > 0 \wedge \lambda > 0$

or $r < 0 \wedge \lambda < 0$

$\dot{r} < 0$ for $r > 0 \wedge \lambda < 0$

or $r < 0 \wedge \lambda > 0$

fixed points: $r = 0$ and $r = \lambda$

$F(\lambda, r) = \lambda r (r - \lambda)^2 \Rightarrow \frac{\partial F}{\partial r} = 3r^2\lambda - 4r\lambda^2 + \lambda^3 \Rightarrow 0$ for $(0,0)$

$\frac{\partial F}{\partial \lambda} = r^3 - 4r^2\lambda + 3r\lambda^2 \Rightarrow 0$ for $(0,0)$ (5)

$\Rightarrow$ there is a transcritical bifurcation point at $(0,0)$.

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4) i) A system $\dot{x}_1 = X_1(x_1, x_2)$, $\dot{x}_2 = X_2(x_1, x_2)$ is a Hamiltonian system

iff $\quad \operatorname{div} \vec{X} = 0$

a) $\operatorname{div} \vec{X} = \frac{\partial x_2}{\partial x_1} + \frac{\partial (x_1 - x_1^3)}{\partial x_2} = 0 \Rightarrow$ Hamiltonian system

b) $\operatorname{div} \vec{X} = 3x_1^2 + k\cos x_2 - 3x_1^2 + 2\cos x_2 \Rightarrow$ for $\underline{k = -2}$ this is also a Hamiltonian system (6)

ii)

$$\left.\begin{aligned} \dot{x}_1 = x_2 = \frac{\partial H}{\partial x_2} &\Rightarrow H(x_1, x_2) = \frac{1}{2}x_2^2 + f(x_1) \\ \dot{x}_2 = x_1 - x_1^3 = -\frac{\partial H}{\partial x_1} &\Rightarrow H(x_1, x_2) = \frac{x_1^4}{4} - \frac{x_1^2}{2} + f(x_2) \end{aligned}\right\} \Rightarrow \underline{H(x_1, x_2) = \frac{x_2^2}{2} + \frac{x_1^4}{4} - \frac{x_1^2}{2} + c}$$

$\Rightarrow$ this is also a potential system with $V(x_1) = \frac{x_1^4}{4} - \frac{x_1^2}{2} + c$

$\frac{\partial V}{\partial x_1} = x_1^3 - x_1 \qquad \frac{\partial^2 V}{\partial x_1^2} = 3x_1^2 - 1 \qquad \Rightarrow x_1 = 0$ is a maximum and $x_1 = \pm 1$ are minima

$\Rightarrow \quad \underline{c = \frac{1}{4}}$ (6)

iii) - fixed points: $\quad x_2 = 0 \quad$ and $\quad x_1 - x_1^3 = 0$

$\Rightarrow \quad \vec{x}_f^{(1)} = (0, 0) \quad , \quad \vec{x}_f^{(2)} = (1, 0) \quad , \quad \vec{x}_f^{(3)} = (-1, 0)$ (2)

- their type is determined by the condition

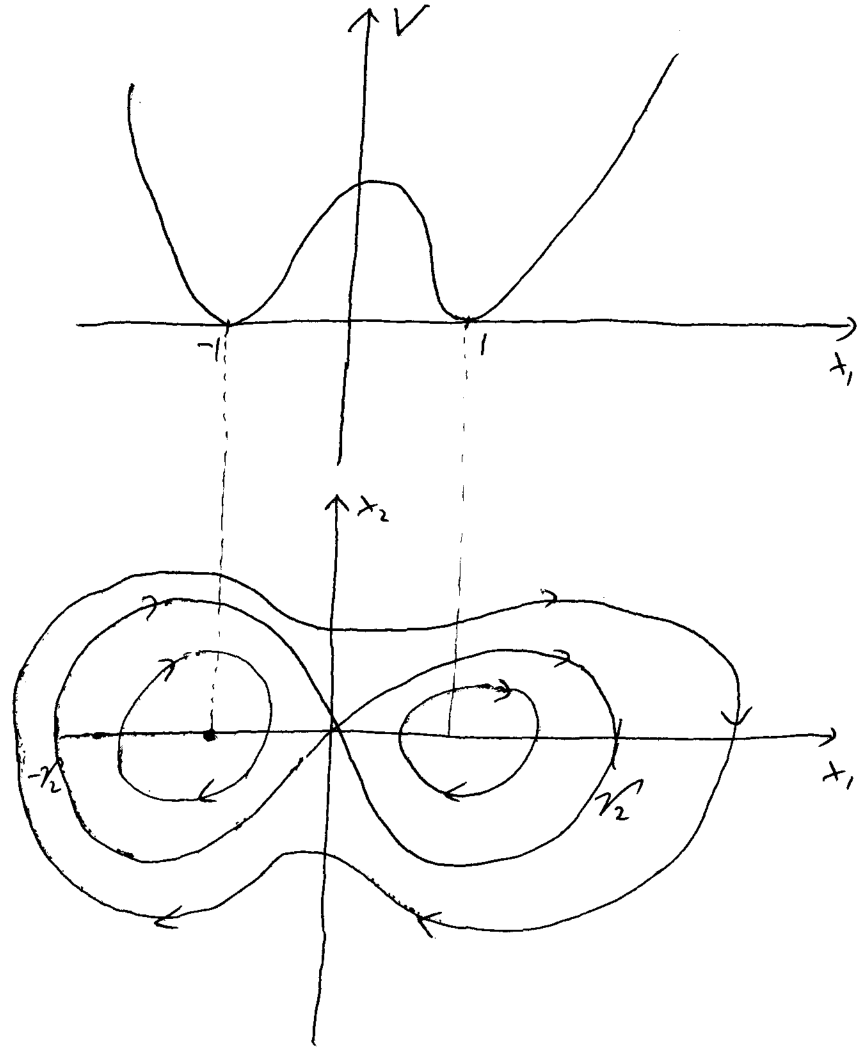
$$H_{12}^2 - H_{11}H_{22} \begin{cases} > 0 & \equiv \text{saddle point} \\ < 0 & \equiv \text{centre} \end{cases}$$

with $H_{11} = 3x_1^2 - 1 \qquad H_{22} = 1 \qquad H_{12} = 0$

$$\Rightarrow \quad H_{12}^2 - H_{11}H_{22} = \begin{cases} -2 & \text{for } \vec{x}_f^{(3)} \\ 1 & \text{for } x_f^{(1)} \\ -2 & \text{for } \vec{x}_f^{(2)} \end{cases}$$

$\Rightarrow \quad \vec{x}_f^{(2,3)}$ are centres and $\vec{x}_f^{(1)}$ is a saddle point. (6)

iv)



(4)

- the seperatrix crosses the saddle point
- $H(0,0) = \frac{1}{4}$, $\because$ H is conserved along trajectories

$\Rightarrow \quad 0 = \frac{x_2^2}{2} + \frac{x_1^4}{4} - \frac{x_1^2}{2} \quad \Rightarrow \quad \underline{x_2 = \pm x_1 \sqrt{1 - \frac{x_1^2}{2}}}$ is the equation of the seperatrix

- directions from $x_2 > 0 \Rightarrow \dot{x}_1 > 0$ and $x_2 < 0 \Rightarrow \dot{x}_1 < 0$. (6)

$\boxed{\Sigma = 30}$

5) i)

$$x_{n+1} = 2\lambda x_n - \lambda x_n^2$$

fixed points: $x_n = 2\lambda x_n - \lambda x_n^2 \Rightarrow x_n^{f_1} = 0$ and $x_n^{f_2} = 2 - \frac{1}{\lambda}$

stability: x_f is a stable fixed point if $|F'(x_f)| < 1$

$\Rightarrow F'(x) = 2\lambda - 2\lambda x = 2\lambda(1-x)$

$|F'(x)| = |2\lambda| > 1$ for $\lambda > \frac{1}{2} \Rightarrow x^f = 0$ is unstable

$|F'(2-\frac{1}{\lambda})| = |2\lambda(\frac{1}{\lambda} - 1)| = |2 - 2\lambda|$ (6)

⇒ x^{*_2} is stable for $|1-\lambda| < \frac{1}{2} \iff -\frac{1}{2} < 1-\lambda < \frac{1}{2}$

$\iff -\frac{3}{2} < -\lambda < -\frac{1}{2} \iff \frac{1}{2} < \lambda < \frac{3}{2}$

ii/ A two cycle exists when $F(F(x)) = x$

$\Rightarrow \quad x = 2\lambda F(x) - \lambda F^2(x)$

$\Rightarrow \quad 0 = 4x\lambda^2 - 2x^2\lambda^2 - 4x^2\lambda^3 + 4x^3\lambda^3 - x^4\lambda^3 - x$

$= -x(1 - 4\lambda^2 + 2x\lambda^2 + 4x\lambda^3 - 4x^2\lambda^3 + x^3\lambda^3)$

This should equal: $(F(x) - x)(1 + 2\lambda - x\lambda - 2x\lambda^2 + x^2\lambda^2) = 0$

Verify this: $-x(1 - 2\lambda + x\lambda)(1 + 2\lambda - x\lambda - 2x\lambda^2 + x^2\lambda^2)$

$= -x(1 + 2\lambda - x\lambda - 2x\lambda^2 + x^2\lambda^2 - 2\lambda - 4\lambda^2 + 2x\lambda^2 + 4x\lambda^3$
$- 2x^2\lambda^3 + x\lambda + 2x\lambda^2 - x^2\lambda^2 - 2x^2\lambda^3 + x^3\lambda^3)$

$\overset{!}{=} -x(1 - 4\lambda^2 + 2x\lambda^2 + 4x\lambda^3 - 4x^2\lambda^3 + x^3\lambda^3)$ ✓ (10)

Solve now: $1 + 2\lambda - x\lambda - 2x\lambda^2 + x^2\lambda^2 = 0 \Rightarrow x_\pm = \frac{1+2\lambda \pm \sqrt{4\lambda^2 - 4\lambda - 3}}{2\lambda}$

For this to be real we need: $4\lambda^2 - 4\lambda - 3 > 0 \Rightarrow \lambda > \frac{3}{2}$ (4)

iii/ the stability condition for a two-cycle is:

$$\left| \frac{d}{dx} F^2(x)\Big|_{x_\pm} \right| = |F'(x_-)F'(x_+)| < 1 \qquad F'(x) = 2\lambda(1-x)$$

$$\Rightarrow F'(x_-)F'(x_+) = 4\lambda^2(1-x_-)(1-x_+) = 4\lambda^2\Big(1 - \underbrace{(x_- + x_+)}_{\frac{1+2\lambda}{\lambda}} + \underbrace{x_- x_+}_{\frac{(1+2\lambda)^2 - (4\lambda^2 - 4\lambda - 3)}{4\lambda^2}}\Big)$$

$$= 4\lambda^2\left(1 - 2 - \frac{1}{\lambda} + \frac{1 + 4\lambda + \cancel{4\lambda^2} - \cancel{4\lambda^2} + 4\lambda + 3}{4\lambda^2}\right)$$

$$= 4\lambda^2\left(-1 - \frac{1}{\lambda} + \frac{4+8\lambda}{4\lambda^2}\right) = 4 + 8\lambda - 4\lambda - 4\lambda^2 = 4 + 4\lambda - 4\lambda^2 = 4(1 + \lambda - \lambda^2)$$

⇒ stable 2 cycle for $4|\lambda^2 - \lambda - 1| < 1$ (10)

⇒ the 2 cycle is stable for $\frac{3}{2} < \lambda < \frac{1}{2}(1+\sqrt{6})$ and unstable for $\lambda > \frac{1}{2}(1+\sqrt{6})$

$\boxed{\Sigma = 30}$