Solutions: Dynamical Systems MA3608 (2007)

1) i) *fixed points*: (1) $8x_1 + x_1 x_2^2 + 3x_2^3 = 0$

(2) $2x_1 x_2 (x_1 + x_2) = 0 \Rightarrow x_1 = -x_2$ or $x_1 = x_2 = 0$

3 $\Rightarrow$ (1) $8x_1 + x_1^3 - 3x_1^3 = 0 \Rightarrow x_1(4 - x_1^2) = 0 \Rightarrow x_1 = \pm 2$

$\Rightarrow x_2 = \mp 2$

$\Rightarrow x_F^{(1)} = (0, 0) \quad x_F^{(2)} = (2, -2) \quad x_F^{(3)} = (-2, 2)$

Linearisation theorem: Let a nonlinear system have a simple linearisation at some fixed point. Then in a

3 neighbourhood of the fixed point the phase portrait of the nonlinear system and the one of its linearisation are qualitatively equivalent if the eigenvalues have a real part.

Jacobian matrix:
$$A(x_1, x_2) = \begin{pmatrix} -8 - x_2^2 & -2x_1 x_2 - 9x_2^2 \\ 4x_1 x_2 + 2x_2^2 & 2x_1^2 + 4x_1 x_2 \end{pmatrix}$$

$$\Rightarrow A(x_F^{(1)}) = \begin{pmatrix} -8 & 0 \\ 0 & 0 \end{pmatrix} \qquad A(X_F^{(2)}) = \begin{pmatrix} -12 & -28 \\ -8 & -8 \end{pmatrix} \qquad A(x_F^{(3)}) = A(x_F^{(2)})$$

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For $x_F^{(1)}$ the linearisation is non-simple as $\det A(x_F^{(1)}) = 0$, such that the LT can not be applied.

For $x_F^{(2)}, x_F^{(3)}$ we have $\det A_\lambda = \begin{vmatrix} -12-\lambda & -28 \\ -8 & -8-\lambda \end{vmatrix} = \lambda^2 + 20\lambda - 128$

$\Rightarrow \lambda_\pm = -10 \pm 2\sqrt{57}$ $\equiv$ positive and negative real eigenvalues

$\Rightarrow$ there are saddle points at $x_F^{(2)}$ and $x_F^{(3)}$

$\Rightarrow$ the LT can be applied.

ii) *Lyapunov stability theorem*: Consider a dynamical

3 system of the form $\dot{\vec{x}} = \vec{X}(\vec{x})$ with a fixed point

at the origin. If there exists a real valued function
$V(\vec{x})$ in a neighbourhood of the origin, such that:

a) the partial derivatives $\partial V/\partial x_1$ and $\partial V/\partial x_2$ exist and are continuous

b) V is positive definite

c) $\dot{V}$ is negative semi-definite (definite)

Then the origin is a stable (asymptotically stable) fixed point.

Ex $V(x_1, x_2) = 2x_1^2 + 3x_2^2$

a) ✓

(3) b) $V(0,0) = 0$ and $V(x_1, x_2) > 0$ for $\vec{x} \neq (0,0) \Rightarrow V$ is positive definite

c) $$\dot{V} = \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2$$

$$= 4x_1(-8x_1 - x_1x_2^2 - 3x_2^3) + 6x_2(2x_1x_2^2 + 2x_2x_1^2)$$

$$= -32x_1^2 - 4x_1^2x_2^2 - 12x_1x_2^3 + 12x_1x_2^3 + 12x_2^2x_1^2$$

$$= -32x_1^2 + 8x_1^2x_2^2 = 8x_1^2(x_2^2 - 4) \leq 0 \quad \text{for } |x_2| < 2$$

$\Rightarrow \dot{V}$ is negative semi-definite for $|x_2| < 2$

$\Rightarrow V$ is a weak Lyapunov function for $|x_2| < 2$

$\Rightarrow$ The origin is a stable fixed point by the LST.

(iii) The extension of the LST states:

Let V be a weak Lyapunov function for the system $\dot{\vec{x}} = \vec{X}(\vec{x})$
(2) in a neighbourhood of an isolated fixed point $\vec{x}_F = (0,0)$.
Then if $\dot{V} \neq 0$ on a trajectory, except for the fixed point itself, the origin is asymptotically stable.

Here we have $\dot{V}(\vec{x}) = 0$ for $\vec{x} = (0, x_2)$. However, on that line we have $\dot{x}_1 = -3x_2^3$, $\dot{x}_2 = 0$, which

2 means $\vec{x} = (0, x_2)$ is not a trajectory.

$\Rightarrow \vec{x} = (0,0)$ is asymptotically stable.

iv) The domain of stability requires $|x_2| < 2$

1 $$\Rightarrow \quad 2x_1^2 + 3x_2^2 < 12$$

is the domain of stability

$\boxed{\sum_1 = 20}$

2) i) Bendixson's criterion: Let D be a simply connected region of the phase space (plane) in which the function $\vec{X}(\vec{x})$ of the system $\dot{\vec{x}} = \vec{X}(\vec{x})$ has the property that its

2 divergence is of constant sign, i.e.

$$\operatorname{div} \vec{X} = \frac{\partial X_1}{\partial x_1} + \frac{\partial X_2}{\partial x_2} \begin{matrix} > 0 \\ < 0 \end{matrix} \text{ or}$$

Then the system has no closed orbit contained entirely in D

1 ii) A limit cycle contains at least one fixed point.

iii) a) The system has no fixed point:

2 $$\left.\begin{aligned} 1 + x_2^2 - e^{x_1 x_2} &= 0 \\ x_1 x_2 + \cos^2(x_2) &= 0 \end{aligned}\right\} \Rightarrow \quad 1 + x_2^2 - e^{-\cos^2(x_2)} = 0$$

$\Rightarrow$ no solution for x_2

$\Rightarrow$ The system has no limit cycle by ii)

b) $$\operatorname{div} \vec{X} = 3x_1^2 + 1 + 3x_2^2 = 1 + 3(x_1^2 + x_2^2) > 0$$

2 $\Rightarrow$ by i) follows there is no limit cycle in the entire $x_1 x_2$-plane.

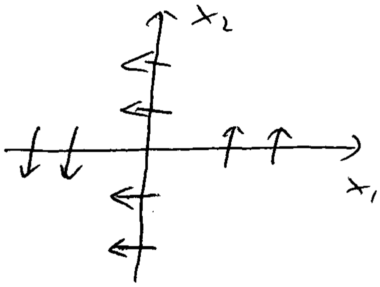
c) $\operatorname{div} \vec{X} = 1 - 3x_1^2 x_2^2 - 1 = -3x_1^2x_2^2 < 0$ for $x_1 \neq \wedge x_2 \neq$

$\Rightarrow$ By i) follows that there is no limit cycle in any quadrant of the x_1x_2-plane

- We could still have a closed orbit crossing the x_1 or x_2 axis

(4) - We have however $\dot{x}_1 = -x_2^2$ for $x_1 = 0$

$\dot{x}_2 = x_1^5$ for $x_2 = 0$



$\Rightarrow$ we can not close any trajectory which goes from one quadrant to another

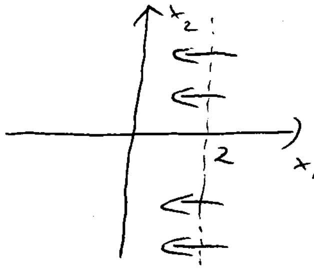
$\Rightarrow$ there is no limit cycle in the entire plane

d) $\operatorname{div} \vec{X} = 4 - 2x_1 + x_1^2 = (x_1 - 2)^2 > 0$ for $x_1 \neq 2$

$\Rightarrow$ By i) follows that there is no limit cycle in the half planes $x_1 > 2$ or $x_1 < 2$

(4) - We have $\dot{x}_1 = -x_2^2$ for $x_1 = 2$, which means we can only cross this line in one direction and can not form a closed orbit

$\Rightarrow$ there is no limit cycle in the entire plane



e) $\operatorname{div} \vec{X} = 2x_1 + x_1 - 2 = 3x_1 - 2 \begin{cases} < 0 & x_1 < 2/3 \\ = 0 & x_1 = 2/3 \\ > 0 & x_1 > 2/3 \end{cases}$

(5) fixed points: (1) $x_1^2 - x_2 - 1 = 0$

(2) $x_2(x_1 - 2) = 0 \Rightarrow x_2 = 0 \vee x_1 = 2$

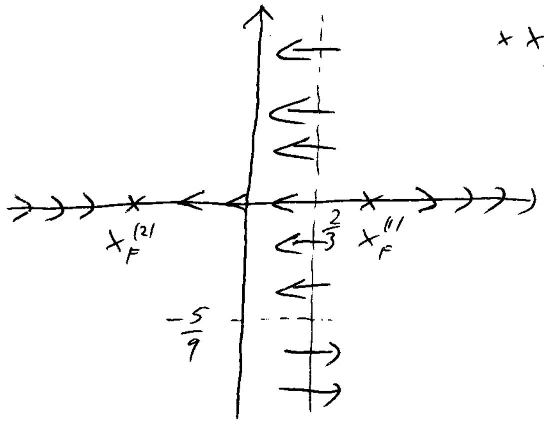
$x_2 = 0$ into (1) $\Rightarrow$ $x_1^2 = 1$ $\Rightarrow$ $x_1 = \pm 1$

$x_1 = 2$ into (1) $\Rightarrow$ $x_2 = 3$

$\Rightarrow$ $x_F^{(1)} = (1, 0)$ $\quad x_F^{(2)} = (-1, 0)$ $\quad x_F^{(3)} = (2, 3)$

We also have $\dot{x}_1 = -\frac{5}{9} - x_2$ for $x_1 = 2/3$

$\dot{x}_2 = 0, \ \dot{x}_1 = x_1^2 - 1$ for $x_2 = 0$



$\times\, x_F^{(3)}$ $\Rightarrow$ We can not close any trajectory around a fixed point

$\Rightarrow$ by ii) no limit cycle in the entire plane

$$\boxed{\Sigma = 20}$$

3) i) $\dot{x}_1 = x_2 + x_1(4 - 5x_1^2 - 5x_2^2)$

(5) $\dot{x}_2 = -x_1 + 5x_2(1 - x_1^2 - x_2^2)$

<u>Jacobian matrix</u>: $A = \begin{pmatrix} 4 & 1 \\ -1 & 5 \end{pmatrix}$ at the origin

$\Rightarrow \det A_\lambda = \begin{vmatrix} 4-\lambda & 1 \\ -1 & 5-\lambda \end{vmatrix} = \lambda^2 - 9\lambda + 21 \Rightarrow \lambda_\pm = \frac{9}{2} \pm \frac{i}{2}\sqrt{3}$

$\Rightarrow$ $\lambda_\pm$ is complex with positive real part

$\Rightarrow$ the linearisation theorem applies

$\Rightarrow$ the origin is an unstable focus

ii) $\dot{r}\cos\vartheta - r\sin\vartheta\,\dot{\vartheta} = r\sin\vartheta + 4r\cos\vartheta\left(1 - \frac{5}{4}r^2\right)$ (1)

$\dot{r}\sin\vartheta + r\cos\vartheta\,\dot{\vartheta} = -r\cos\vartheta + 5r\sin\vartheta(1 - r^2)$ (2)

(5) (1) $\cos\vartheta$ + (2) $\sin\vartheta$: $\dot{r} = r(4 + \sin^2\vartheta - 5r^2)$

(2) $\cos\vartheta$ − (1) $\sin\vartheta$: $r\dot{\vartheta} = -r + 5r\sin\vartheta\cos\vartheta(1 - r^2) - 4\sin\vartheta\cos\vartheta\left(1 - \frac{5}{4}r^2\right)$

$\Rightarrow \dot{\vartheta} = \frac{1}{2}\sin 2\vartheta - 1 < 0 \quad \forall r, \vartheta \Rightarrow$ no other fixed point

iii) Poincaré - Bendixson theorem: Let φ_t be a flow for the system $\dot{\vec{x}} = \vec{X}(\vec{x})$ and let D be a closed,

(5) bounded and connected set $D \subset \mathbb{R}^2$ such that $\varphi_t(D) \subset D \;\forall t$. In addition D does not contain any fixed point. Then there exists at least one limit cycle in D.

At $r = \frac{1}{2}$: $\quad \dot r = \frac{1}{2}\left(4 + \sin^2\vartheta - \frac{5}{4}\right) > 0$

$r = 2$: $\quad \dot r = 2\left(4 + \sin^2\vartheta - 5\cdot 4\right) < 0$

$\Rightarrow$ trajectories which enter $D = \{(r, \vartheta) : \frac{1}{2} \le r \le 2\}$ do not leave it anymore

$\Rightarrow$ by the PBT it follows that there exists at least one limit cycle in D

iv) - $\dot r > 0 \iff 4 + \sin^2\vartheta - 5r^2 > 0 \iff r^2 < \frac{4 + \sin^2\vartheta}{5}$

(5) on the inner boundary we can make r smaller without destroying the picture $\Rightarrow r < \sqrt{\min\left(\frac{4+\sin^2\vartheta}{5}\right)} = \sqrt{\frac{4}{5}}$

- $\dot r < 0 \iff 4 + \sin^2\vartheta - 5r^2 > 0 \iff r^2 > \frac{4 + \sin^2\vartheta}{5}$

on the outer boundary we can make r larger without destroying the picture $\Rightarrow r > \sqrt{\max\left(\frac{4+\sin^2\vartheta}{5}\right)} = 1$

- In $D^\varepsilon = \{(r, \vartheta) : \sqrt{4/5} - \varepsilon \le r \le 1 + \varepsilon\}$ we have no fixed points

- $\dot r > 0$ on inner boundary and $\dot r$ 0 on outer boundary means that trajectories which enter D^ε never leave D^ε

- $\because$ $r = \sqrt{4/5}$ and $r = 1$ are not trajectories the same holds for $D = \{(r, \vartheta) : \sqrt{\frac{4}{5}} \le r \le 1\}$ $\quad \boxed{\Sigma = 20}$

4/ i) A system $\dot{x}_1 = X_1(x_1, x_2)$, $\dot{x}_2 = X_2(x_1, x_2)$

is a Hamiltonian system if and only if

(4)

$$\operatorname{div} \vec{X} = 0$$

a/ $\left.\begin{array}{l}\dot{x}_1 = x_2 \\ \dot{x}_2 = x_1 + x_1^2\end{array}\right\} \Rightarrow \operatorname{div} \vec{X} = 0 + 0 = 0 \Rightarrow$ a) is a Hamilto[nian] system

b/ $\left.\begin{array}{l}\dot{x}_1 = x_2 - x_2^2 + x_1^2 \\ \dot{x}_2 = -x_1 - \lambda x_1 x_2\end{array}\right\} \Rightarrow \operatorname{div} \vec{X} = 2x_1 - \lambda x_1$

$\Rightarrow$ b) is a Hamiltonian system for $\lambda = 2$

ii) For a Hamiltonian system we have

(2)

$$\dot{x}_1 = \frac{\partial H}{\partial x_2} \qquad \dot{x}_2 = -\frac{\partial H}{\partial x_1}$$

$$\Rightarrow \frac{dH}{dt} = \frac{\partial H}{\partial x_1}\dot{x}_1 + \frac{\partial H}{\partial x_2}\dot{x}_2 = \frac{\partial H}{\partial x_1}\frac{\partial H}{\partial x_2} - \frac{\partial H}{\partial x_2}\frac{\partial H}{\partial x_1} = 0$$

$\Rightarrow$ The trajectories are of the form $H(x_1, x_2) = E = \text{const}$

iii)

(2)

$$\dot{x}_1 = \frac{\partial H}{\partial x_2} = x_2 \Rightarrow H(x_1, x_2) = \frac{1}{2}x_2^2 + f(x_1)$$

$$\dot{x}_2 = -\frac{\partial H}{\partial x_1} = x_1 + x_1^2 \Rightarrow H(x_1, x_2) = -\left(\frac{1}{2}x_1^2 + \frac{x^3}{3}\right) + \tilde{f}(x_2)$$

$$\Rightarrow H(x_1, x_2) = \frac{1}{2}x_2^2 - \frac{1}{2}x_1^2 - \frac{1}{3}x_1^3 + C = \frac{1}{2}x_2^2 + V(x_1)$$

From $V(0) = 0 \Rightarrow C = 0$

iv)

(5)

$$V(x_1) = -\frac{1}{2}x_1^2 - \frac{1}{3}x_1^3 \qquad V'(x_1) = 0 \Rightarrow x_1 = 0, -1$$

$$V'(x_1) = -x_1 - x_1^2 \qquad V''(0) = -1 \Rightarrow \text{maximum}$$

$$V''(x_1) = -1 - 2x_1 \qquad V''(-1) = 1 \Rightarrow \text{minimum}$$

The fixed points of the Hamiltonian system

$$H(x_1, x_2) = \frac{1}{2} x_2^2 + V(x_1)$$

are located at $(a_k, 0)$ for $k = 1, 2, \ldots$ where a_k is a stationary point of $V(x_1)$. If $V(a_k)$ is a minimum then the point $(a_k, 0)$ is a centre and if $V(a_k)$ is a maximum then the point $(a_k, 0)$ is a saddle point.

$\Rightarrow$ $(0, 0)$ is a saddle point

$(-1, 0)$ is a centre

v)

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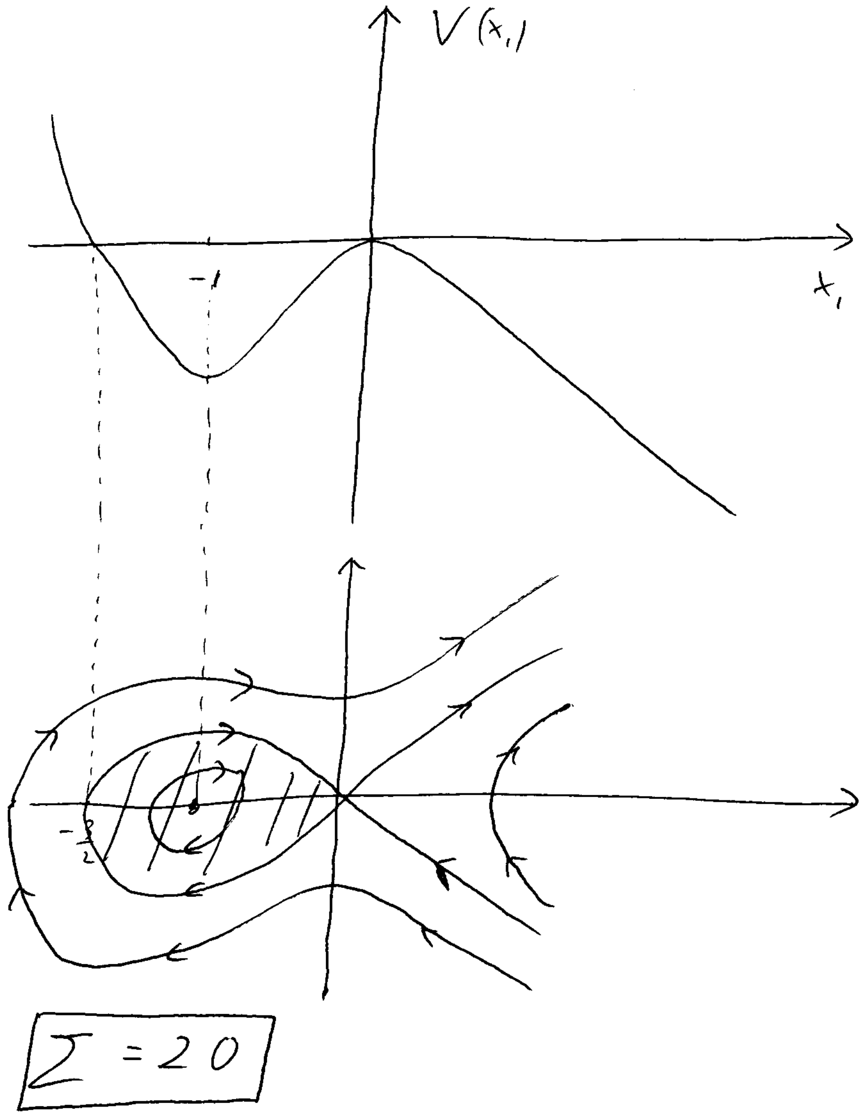


- separatrix:

E at local maximum is zero

$$H(x_1, x_2) = \frac{1}{2} x_2^2 - \frac{1}{2} x_1^2 - \frac{1}{3}$$

$$= 0$$

$$\Rightarrow x_2 = \pm \sqrt{x_1^2 + \frac{2}{3} x_1^3}$$

$$\Rightarrow x_2 = 0 \text{ for } x_1 = \frac{-3}{2}$$

- /// is the bounded region

- the time direction follows from $\dot{x}_1 = x_2$

$$\boxed{\Sigma = 20}$$

5) i) fixed points: $F(x) = x \iff \lambda x - \lambda x^2 = x$

5

$$\Rightarrow x(\lambda x + 1 - \lambda) = 0 \quad \Rightarrow \quad x_f^{(1)} = 0$$

$$\Rightarrow \quad x_f^{(2)} = 1 - \frac{1}{\lambda}$$

A fixed point is stable if $|F'(x_f)| < 1$.

Here $F'(x) = \lambda - 2x\lambda$

$\Rightarrow F'(x_f^{(1)}) = \lambda \quad \Rightarrow x_f^{(1)}$ is stable for $\lambda < 1$

unstable for $\lambda > 1$

$$F'(x_f^{(2)}) = \lambda - 2\lambda + 2\,\frac{\lambda}{\lambda} = 2 - \lambda$$

$$\Rightarrow |2-\lambda| < 1 \quad \Leftrightarrow \quad (2-\lambda)^2 < 1$$

$$\underbrace{\lambda^2 - 4\lambda + 3}_{(\lambda-1)(\lambda-3)} < 0$$

$x_f^{(2)}$ is stable for $1 < \lambda < 3$

$x_f^{(2)}$ is unstable for $\lambda > 3$

ii) A 2-cycle is defined by the condition $F^2(x) = x$.

(5) With $F(x) = \lambda x - \lambda x^2$

$$\Rightarrow \lambda F(x) - \lambda \left(F(x)\right)^2 = x$$

$$\lambda^2 x - \lambda^2 x^2 - \lambda(\lambda x - \lambda x^2)^2 = x$$

$$\Leftrightarrow x(\lambda^2 - 1) - x^2(\lambda^2 + \lambda^3) + 2\lambda^3 x^3 - \lambda^3 x^4 = 0$$

$$-\underbrace{x(\lambda x + 1 - \lambda)}_{\text{fixed point}} \underbrace{\left(x^2\lambda^2 - x(\lambda^2 + \lambda) + 1 + \lambda\right)}_{\text{two cycle}} = 0$$

$$\Rightarrow x_{1/2} = \frac{1}{2\lambda}\left(1 + \lambda \pm \sqrt{\underbrace{\lambda^2 - 2\lambda - 3}_{(1+\lambda)(\lambda-3)}}\right)$$

for $\lambda \geq 3$ these solutions are real, such that a 2 cycle $\exists$

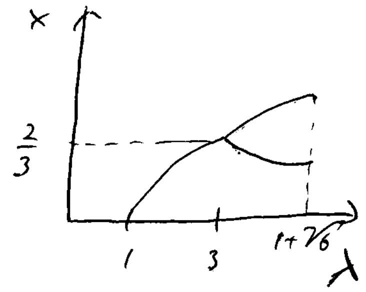
iii) The 2 cycle is stable if for $G(x) = F^2(x)$

(5) $|G'(x)| < 1 \quad \Leftrightarrow \quad |F'(x_1)\, F'(x_2)| < 1$

$$\Rightarrow \left|\left[\lambda - (1 + \lambda + \sqrt{\cdot})\right]\left[\lambda - (1 + \lambda - \sqrt{\cdot})\right]\right| = \left|(-1 + \sqrt{\cdot})(-1 - \sqrt{\cdot})\right|$$

$$= |1 - (\lambda^2 - 2\lambda - 3)| = |-\lambda^2 + 2\lambda + 4| < 1 \quad \Rightarrow$$ $3 < \lambda < 1 + \sqrt{6}$ is the domain of

$\Rightarrow \; 3 < \lambda < 1+\sqrt{6}$ is the domain of stability



bifurcation diagram

iv) $\quad \lambda = 4 \qquad x = \sin^2 \vartheta$

(5) $\Rightarrow \; F(\vartheta) = 4\sin^2\vartheta - 4\sin^4\vartheta = 4\sin^2\vartheta\underbrace{(1-\sin^2\vartheta)}_{\cos^2\vartheta} = \sin^2(2$

$\Rightarrow$ The difference equation becomes

$$\sin^2\vartheta_{n+1} = \sin^2 2\vartheta_n$$

$$\Rightarrow \quad \vartheta_{n+1} = \begin{cases} 2\vartheta_n & \text{for } 0 \le \vartheta_n \le \frac{\pi}{4} \\ \pi - 2\vartheta_n & \text{for } \frac{\pi}{4} \le \vartheta_n \le \frac{\pi}{2} \end{cases}$$

$$\Rightarrow \quad \theta_{n+1} = \begin{cases} 2\theta_n & \text{for } 0 \le \theta_n \le \frac{1}{2} \\ 2(1-\theta_n) & \text{for } \frac{1}{2} \le \theta_n \le 1 \end{cases} \qquad \text{with } \theta = \pi\vartheta$$

which is the tent map.

$$\boxed{\Sigma = 20}$$