Dynamical Systems II (Exam 07/08)

1) i) $\varepsilon_{1/2}$ measures the influence of species 2/1 on species 1/2. When $\varepsilon_1 = \varepsilon_2 = 0$ the two systems decouple into two logistic
(1) equations. Each species grows until it reaches its saturation level.

ii)

$$\dot{x}_1 = x_1(1 - x_1 - x_2)$$

$$\dot{x}_2 = x_2\left(\tfrac{3}{2} - x_2 - 2x_1\right)$$

fixed points: (1) $x_1(1 - x_2 - x_2) = 0$ (2) $x_2\left(\frac{3}{2} - x_2 - 2x_1\right) = 0$

$x_1 = 0$ into (2) $\frac{3}{2} - x_2 = 0 \Rightarrow x_2 = \frac{3}{2}$

$x_2 = 0$ int (1) $1 - x_1 = 0 \Rightarrow x_1 = 1$

$1 - x_1 - x_2 = 0 \wedge \frac{3}{2} - x_2 - 2x_1 = 0 \Rightarrow \frac{3}{2} - 1 + x_1 - 2x_1 = 0 \Rightarrow x_1 = \frac{1}{2} \Rightarrow x_2 = \frac{1}{2}$

$\Rightarrow$ The fixed points are:

(4) $\vec{x}_F^{(1)} = (0,0)$ $\quad x_F^{(2)} = (0, \frac{3}{2})$ $\quad x_F^{(3)} = (1, 0)$ $\quad x_F^{(4)} = (\frac{1}{2}, \frac{1}{2})$

iii) In $x_F^{(1)}$ both species are extinct.

(1) In $x_F^{(2)}$, $x_F^{(3)}$ one of the two species is extinct.

In $x_F^{(4)}$ both species can coexist.

iv) Linearisation theorem:

Consider a non-linear system with a simple linearisation at some fixed point. Then in a neighbourhood of the fixed point the phase portrait of the non-linear system and its linearisation are qualitatively equivalent
(2) if the eigenvalues of the Jacobian matrix have a nonzero real part, i.e. the linearisation is not a centre.

Jacobian matrix:

$$A(x_1, x_2) = \begin{pmatrix} 1 - 2x_1 - x_2 & -x_1 \\ -2x_2 & \frac{3}{2} - 2x_1 - 2x_2 \end{pmatrix}$$

$A(x_F^{(1)}) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{3}{2} \end{pmatrix} \Rightarrow$ C.E.: $\lambda^2 - \frac{5}{2}\lambda + \frac{3}{2} = 0 \Rightarrow \lambda_1 = \frac{3}{2},\ \lambda_2 = 1 \equiv$ unstable node

$A(x_F^{(2)}) = \begin{pmatrix} -\frac{1}{2} & 0 \\ -3 & -\frac{3}{2} \end{pmatrix} \Rightarrow$ C.E.: $\lambda^2 + 2\lambda + \frac{3}{4} = 0 \Rightarrow \lambda_1 = -\frac{1}{2},\ \lambda_2 = -\frac{3}{2} \equiv$ stable node

$A(x_F^{(3)}) = \begin{pmatrix} -1 & -1 \\ 0 & -\frac{1}{2} \end{pmatrix} \Rightarrow$ C.E.: $\lambda^2 + \frac{3}{2}\lambda + \frac{1}{2} = 0 \Rightarrow \lambda_1 = -1,\ \lambda_2 = -\frac{1}{2} \equiv$ stable node

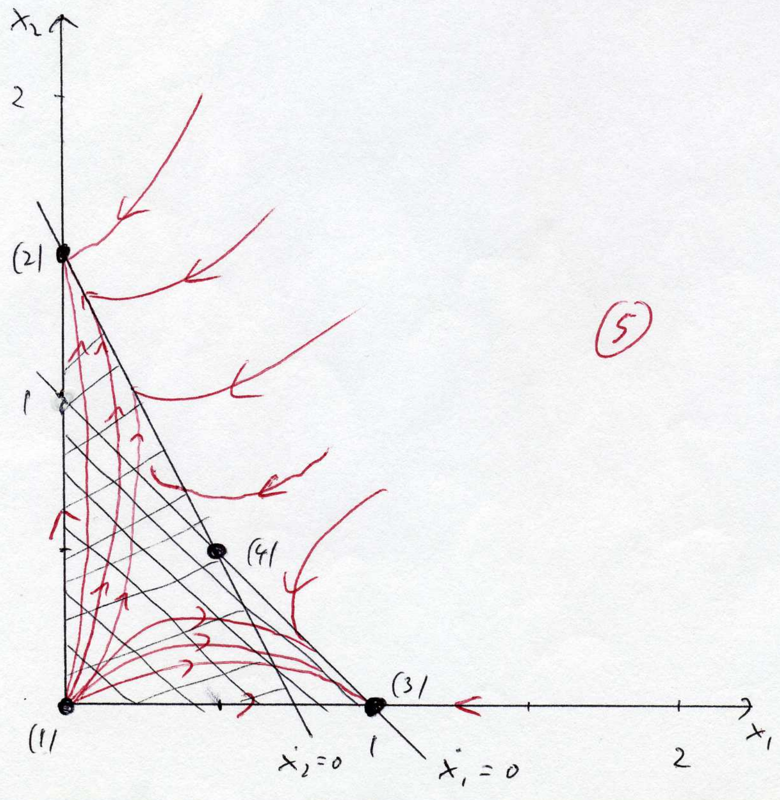
(4) $A(x_F^{(4)}) = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} \\ -1 & -\frac{1}{2} \end{pmatrix} \Rightarrow$ C.E.: $\lambda^2 + \lambda - \frac{1}{4} = 0 \Rightarrow \lambda_{1/2} = -\frac{1}{2} \pm \frac{1}{\sqrt{2}} \equiv$ saddle point

(1) The linearisation theorem can be applied at each of the fixed points.

v) Isoclines:

$$\frac{dx_2}{dx_1} = \frac{\dot{x}_2}{\dot{x}_1} = \frac{(\frac{3}{2} - x_2 - 2x_1)x_2}{x_1(1 - x_1 - x_2)} \begin{cases} \to 0 & \text{for } x_2 = \frac{3}{2} - 2x_1 \\ \to \infty & \text{for } x_2 = 1 - x_1 \end{cases}$$

(2)



(5)

$\dot{x}_2 > 0$:

$x_2 > 0 \ \wedge\ \frac{3}{2} - x_2 - 2x_1 > 0$

$\Rightarrow x_2 < \frac{3}{2} - 2x_1$

$x_2 < 0$ is irrelevant

$\dot{x}_1 > 0$:

$x_1 > 0 \ \wedge\ 1 - x_1 - x_2 > 0$

$\Rightarrow x_2 < 1 - x_1$

$x_1 < 0$ is irrelevant

$\Sigma = 20$

2) i) Lyapunov stability theorem:

Consider the dynamical system $\dot{\vec{x}} = \vec{X}(\vec{x})$ with a fixed point at the origin. If there exists a real valued function $V(\vec{x})$ in a neighbourhood $N(\vec{x}=0)$ such that

(a) the partial derivatives $\frac{\partial V}{\partial x_1}$, $\frac{\partial V}{\partial x_2}$ exist and are continuous

(b) V is positive definite

(c) dV/dt is negative semi-definite (definite)

then the origin is a stable (asymptotically stable) fixed point.

(3)

Def.: A function V for which (a) - (b) holds with (c) semi-definite (definite) is called a weak (strong) Lyapunov function.

(1)

- (a) clearly the derivatives exist and are continuous

(b) $\because$ $V(0,0) = 0$ and $V(x_1, x_2) > 0$ for $(x_1, x_2) \neq 0$

$\Rightarrow$ $V(x_1, x_2)$ is positive definite

(c)
$$\dot{V} = \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2$$
$$= 16x_1\left(-3x_1 - \tfrac{1}{2}x_2^3 + x_1x_2^2\right) + \left(2x_1x_2^2 + 2x_2x_1^2\right)4x_2$$
$$= -48x_1^2 - 8x_1x_2^3 + 16x_1^2x_2^2 + 8x_1x_2^3 + 8x_2^2x_1^2$$
$$= -48x_1^2 + 24x_1^2x_2^2$$
$$= 24x_1^2(x_2^2 - 2)$$

(6)

$\Rightarrow$ $\dot{V} = 0$ for $(0, x_2)$

$\Rightarrow$ $\dot{V} \leq 0$ for $|x_2| < \sqrt{2}$, i.e. V is negative semi-definite

$\Rightarrow$ V is a weak Lyapunov function

(3) $\Rightarrow$ by LST follows that the origin is a stable fixed point

ii) For $|x_2| < \sqrt{2}$ all points inside the level curve of $V(x_1, x_2) = 8x_1^2 + 2x_2^2$ will be dragged to the origin

$$\Rightarrow \quad 8x_1^2 + 2x_2^2 < 4$$

$$\Rightarrow \quad 2x_1^2 + \frac{x_2^2}{2} < 1$$

$\Rightarrow$ The length of the minor and major axis of the ellipse which confines the domain of stability is

(4) $2/\sqrt{2}$ and $2 \cdot \sqrt{2}$, respectively.

iii) Let $V(\vec{x})$ be a weak Lyapunov function for the system $\dot{\vec{x}} = \vec{X}(\vec{x})$ in a neighbourhood of an isolated fixed point $\vec{x}_f = (0,0)$. Then, if $\dot{V} \neq 0$ on a trajectory except for the fixed point, the origin is asymptotically stable.

for $(0, x_2)$: $\quad \dot{x}_1 = -\frac{1}{2} x_2^3 \qquad \dot{x}_2 = 0$

$\Rightarrow$ This means the line $(0, x_2)$ is not a trajectory

(3) $\Rightarrow$ The origin is asymptotically stable.

$\Sigma = 20$

3) i) $\dot r \cos\vartheta - r\sin\vartheta\,\dot\vartheta = 25 r\cos\vartheta - r\sin\vartheta - r\cos\vartheta\, r^2$ (1)

$\dot r \sin\vartheta + r\cos\vartheta\,\dot\vartheta = r\cos\vartheta + 25 r\sin\vartheta - r\sin\vartheta\, r^2$ (2)

$\cos\vartheta\,(1) + \sin\vartheta\,(2)$:

$$\dot r(\sin^2\vartheta + \cos^2\vartheta) = 25 r(\cos^2\vartheta + \sin^2\vartheta) - r^3(\cos^2\vartheta + \sin^2\vartheta)$$

(4)

$$\underline{\dot r = r(25 - r^2)}$$

$\sin\vartheta\,(1) - \cos\vartheta\,(2)$:

$$-r\dot\vartheta = -r \quad \Rightarrow \quad \underline{\dot\vartheta = 1} \quad \text{for } r \neq 0$$

(1) $\because \dot\vartheta \neq 0$ the only fixed point is $r = 0$

ii) Poincaré-Bendixson theorem: Let φ_t be a flow for the system $\dot{\vec x} = \vec X(\vec x)$ and let D be a closed bounded and connected set $D \subset \mathbb{R}^2$ such that $\varphi_t(D) \subset D \;\forall t$. In addition D does not contain any fixed point.

(2)

Then there exists at least one limit cycle in D

At $r = 4$: $\quad \dot r = 4(25 - 16) = 36 > 0$

$r = 6$: $\quad \dot r = 6(25 - 36) = -66 < 0$

$\Rightarrow$ trajectories which enter $D = \{(r,\vartheta) : 4 \leq r \leq 6\}$ do not leave it anymore

$\Rightarrow$ there is no fixed point in D

$\Rightarrow$ by PBT it follows that there exists at least one limit cycle in D.

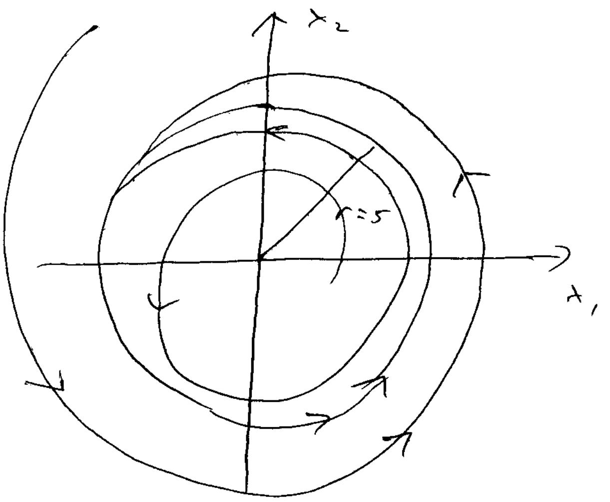
(3)

iii) The limit cycle is at $r=5$; $\dot{r}=0$

$\dot{r}<0$ for $r>0$

$\dot{r}>0$ for $r<5$

~~The limit cycle is stable.~~



(4)

iv) The α-limit set (resp. ω) of a point $\vec{x}$ contains those points which are approached by the trajectory through $\vec{x}$ as $t \to \mp\infty$, that is $L_\alpha(\vec{x}) = \{\vec{y} \in \mathbb{R}^2$: there exists a sequence of times t_n with $t_n \to \mp\infty$ such that $\lim_{n\to\infty} \varphi_{t_n}(\vec{x}) = \vec{y}\}$.

· α-limit set:

$$L_\alpha(\vec{x}) = \begin{cases} x_f = (0,0) & \text{for } r<5 \\ \mathcal{C}_{r=5} & \text{for } r=5 \\ \emptyset & \text{for } r>5 \end{cases}$$

· ω-limit set:

$$L_\omega(\vec{x}) = \begin{cases} x_f = (0,0) & \text{for } r=0 \\ \mathcal{C}_{r=5} & \text{for } r\neq 0 \end{cases}$$

$\Rightarrow$ The limit cycle is stable since $\mathcal{C}_{r=5} = L_\omega(\vec{x})$ for all $\vec{x}$ in some neighbourhood of the limit cycle.

(4)

v) Bendixson's criterion: Let D be a simply connected region of the phase plane in which the function $\vec{X}(\vec{x})$ of the system $\dot{\vec{x}} = \vec{X}(\vec{x})$ has the property that the divergence is of constant sign, i.e. $\operatorname{div}\vec{X} > 0$ or <0.

(2)

Then the system has no closed orbit entirely in D.

$\boxed{\Sigma = 20}$

4/ i) A system $\dot{x}_1 = X_1(x_1, x_2)$, $\dot{x}_2 = X_2(x_1, x_2)$ is a Hamiltonian system if and only if

(1)

$$\operatorname{div} \vec{X} = \frac{\partial X_1}{\partial x_1} + \frac{\partial X_2}{\partial x_2} = 0 .$$

ii) equation of motion:

$$\dot{x}_1 = \frac{\partial H}{\partial x_2} = x_2$$

(2)

$$\dot{x}_2 = -\frac{\partial H}{\partial x_1} = -\frac{1 - 3x_1^4}{(1 + x_1)^4} = \frac{3x_1^4 - 1}{(1 + x_1^4)^2}$$

iii)

$$V(x) = \frac{x}{1 + x^4}$$

$$V'(x) = \frac{1 - 3x^4}{(1 + x^4)^2}$$

$$V''(x) = \frac{4x^3(-5 + 3x^4)}{(1 + x^4)^3}$$

The fixed points of a potential system are located at $(a_k, 0)$ $k = 1, 2, 3, \ldots$ where a_k is a stationary point of $V(x)$. If $V(a_k)$ is a minimum, then the point $(a_k, 0)$ is a centre. If $V(a_k)$ is a maximum then the point $(a_k, 0)$ is a saddle point.

$$V'(x) = 0 \Rightarrow 1 = 3x^4 \Rightarrow x_\pm = \pm 3^{-\frac{1}{4}}$$

$\Rightarrow V''(x_+) < 0$ $\Rightarrow$ maximum at $x_+ = \frac{1}{3^{\frac{1}{4}}}$

$\Rightarrow$ saddle point at $(x_+, 0)$

(6)

$\Rightarrow V''(x_-) > 0$ $\Rightarrow$ minimum at $x_- = -\frac{1}{3^{\frac{1}{4}}}$

$\Rightarrow$ centre at $(x_-, 0)$

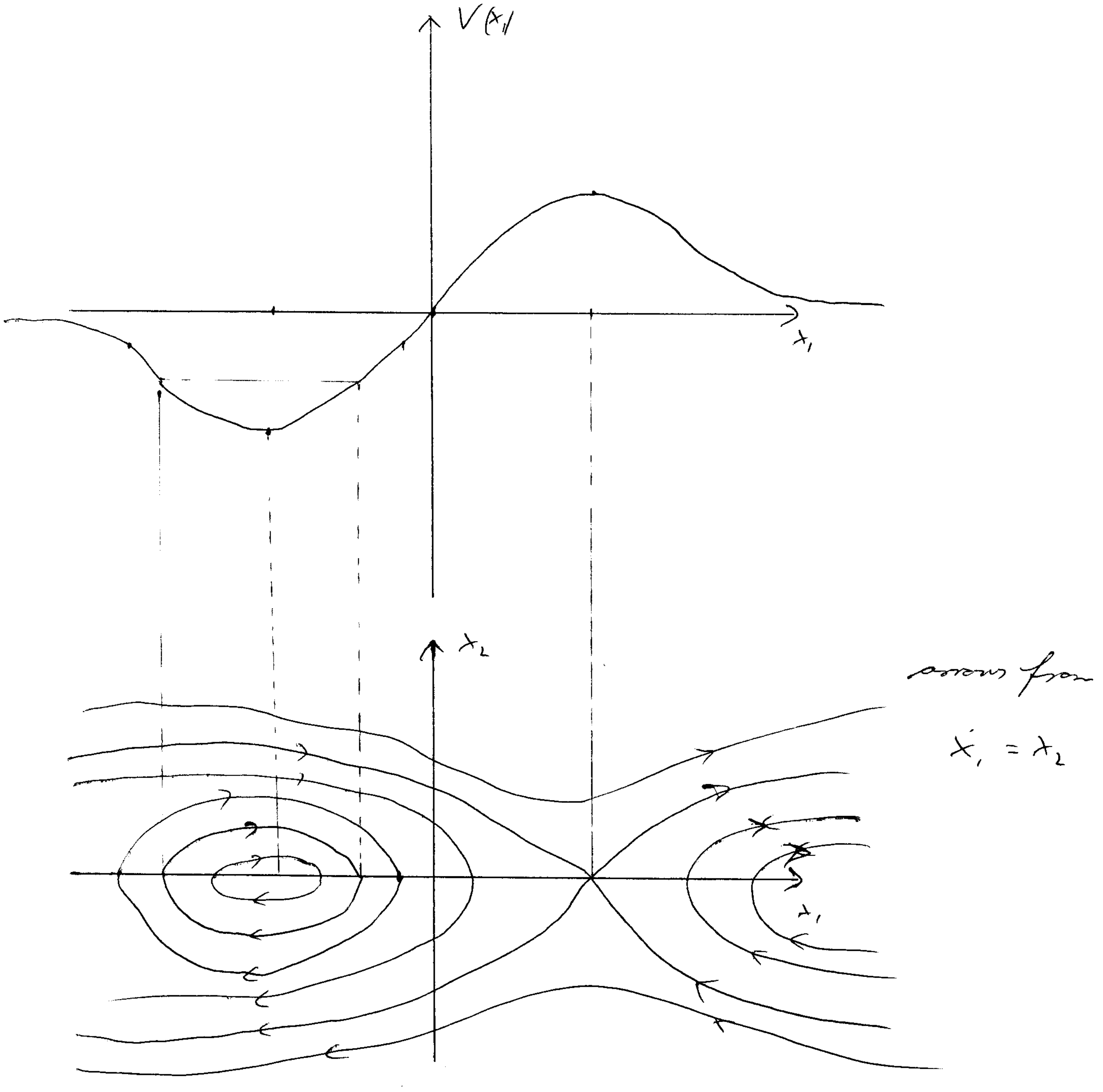
iv) separatrix: $H(x_1, x_2) = H(x_+, 0)$

$$\Rightarrow \frac{1}{2} x_2^2 + \frac{x_1}{1 + x_1^4} = \frac{1}{3^{\frac{1}{4}}} \cdot \frac{1}{4}$$

(2)

$$\Rightarrow x_2 = \pm \sqrt{\frac{1}{2 \cdot 3^{\frac{1}{4}}} - \frac{2x_1}{1 + x_1^4}}$$

Note $\lim_{x \to \pm\infty} V(x) = 0$



(6)

v) The motion is periodic for $V(x) < H(x_1, x_2) = E_p < 0$

$\Rightarrow$ The period is given by

$$T = \oint_{\mathcal{C}} dt = \oint_{c} \frac{dx_1}{\dot{x}_1} = \oint \frac{dx_1}{x_2} = 2\int_{\alpha_1}^{\alpha_2} \frac{dx_1}{\sqrt{2\left(E_p - \frac{x_1}{1+x_1^4}\right)}}$$

(3)

$\Sigma = 20$

5) i) $x_{n+1} = 4\lambda x_n - \lambda x_n^2 = F(x_n)$

fixed points: $x_n = 4\lambda x_n - \lambda x_n^2 \iff x_n\left(x_n + \frac{1}{\lambda} - 4\right) = 0$

(1) $\Rightarrow \underline{x_f^{(1)} = 0} \qquad \underline{x_f^{(2)} = 4 - \frac{1}{\lambda}}$

stability: x_f is a stable fixed point if $|F'(x_f)| < 1$

$\Rightarrow F'(x) = 4\lambda - 2\lambda x$

$\Rightarrow |F'(x_f = 0)| = |4\lambda| > 1 \quad$ for $\lambda > \frac{1}{4}$

$\Rightarrow$ $\underline{x_f = 0 \text{ is unstable for } \lambda > \frac{1}{4}}$

$|F'(4 - \frac{1}{\lambda})| = |4\lambda - 8\lambda + 2| = |2 - 4\lambda|$

$\Rightarrow -1 < 2(1-2\lambda) < 1$

$\Rightarrow \underline{\frac{1}{4} < \lambda < \frac{3}{4}}$

(4) $\Rightarrow x_f = 4 - \frac{1}{\lambda}$ is stable for $\frac{1}{4} < \lambda < \frac{3}{4}$

ii) A two cycle exists if $F(F(x)) = x$

$\Rightarrow x = 4\lambda F(x) - \lambda F^2(x)$

$\Rightarrow x = 16x\lambda^2 - 4x^2\lambda^2 - 16x^2\lambda^3 + 8x^3\lambda^3 - x^4\lambda^3$

Verify that (or by polynomial division

$$(F(x) - x)\underbrace{\left(1 + 4\lambda - x\lambda - 4x\lambda^2 + x^2\lambda^2\right)}_{0} = 0$$

(5) $\Rightarrow x_\pm = \dfrac{4\lambda^2 + \lambda \pm \lambda\sqrt{16\lambda^2 - 8\lambda - 3}}{2\lambda^2}$

This is real if $16\lambda^2 - 8\lambda - 3 \geq 0$

$\Rightarrow$ The existence of the two-cycle requires

$(1 + 4\lambda)(-3 + 4\lambda) \geq 0 \quad \Rightarrow \underline{\lambda \geq \frac{3}{4}}$

(2)

iii) The 2-cycle is stable if for $G(x) = F^2(x)$

$$|G'(x)| < 1 \iff |F'(x_+)\,F'(x_-)| < 1$$

$$\Rightarrow \left|(4\lambda - 2\lambda x_+)(4\lambda - 2\lambda x_-)\right|$$

$$= \left|\left[4\lambda - (4\lambda + 1 + \sqrt{(\;)(\;)})\right]\left[4\lambda - (4\lambda + 1 - \sqrt{(\;)(\;)})\right]\right|$$

$$= \left|(-1 + \sqrt{\;})(-1 - \sqrt{(\;)})\right| = \left|1 - (16\lambda^2 - 8\lambda - 3)\right|$$

$$= \left|16\lambda^2 - 8\lambda - 4\right| < 1$$

$$16\lambda^2 - 8\lambda - 5 < 0 \quad \wedge \quad 16\lambda^2 - 8\lambda - 3 > 0$$

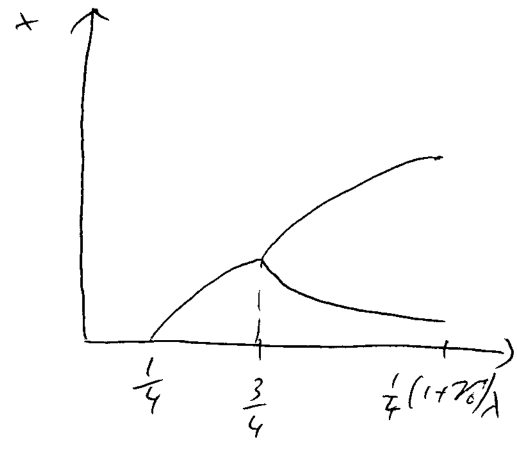
$$\left(\lambda - \tfrac{1}{4}(1 - \sqrt{6})\right)\left(\lambda - \tfrac{1}{4}(1 + \sqrt{6})\right) < 0 \qquad \left(\lambda + \tfrac{1}{4}\right)\left(\lambda - \tfrac{3}{4}\right) > 0$$

(7) $\lambda < \frac{1}{4}(1 + \sqrt{6}) \qquad \lambda > \frac{3}{4}$

$\Rightarrow$ The domain of stability is $\frac{3}{4} < \lambda < \frac{1}{4}(1 + \sqrt{6})$

Bifurcation diagram

(1)



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