MA3608 Dynamical Systems II

Solutions and marking scheme for exam January 2009

INSTRUCTIONS: Full marks are obtained for correct answers to three of the five questions. Each question carries 20 marks.

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1. We have

$$\dot{x}_1 = x_1 x_2^2 - 9x_1 - 16x_2^3$$
$$\dot{x}_2 = 4x_1 x_2^2 + 2x_2 x_1^2$$

(i) **Lyapunov stability theorem:** *Consider the system $\dot{\vec{x}} = \vec{X}(\vec{x})$ with a fixed point at the origin. If there exists a real valued function $V(\vec{x})$ in a neighbourhood $N(\vec{x} = 0)$ such that:*

a) *the partial derivatives $\partial V/\partial x_1$, $\partial V/\partial x_2$ exist and are continuous*

b) *the function $V(\vec{x})$ is positive definite*

c) *dV/dt is negative semi-definite (definite)*

then the origin is a stable (asymptotically stable) fixed point. 3

Def.: *A function V for which the conditions i)-iii) hold with iii) semi-definite is called weak Lyapunov function.*

Def.: *A function V for which the conditions i)-iii) hold with iii) definite is called strong Lyapunov function.* 1

Verify the requirements a)-c) for $V(x_1, x_2) = 4x_1^2 + 16x_2^2$

a) Clearly the partial derivatives $\partial V/\partial x_1$, $\partial V/\partial x_2$ exist and are continuous.

b) $\because V(0,0) = 0$ and $V(x_1, x_2) > 0$ for $(x_1, x_2) \neq 0$
$\Rightarrow$ the function $V(x_1, x_2) = 4x_1^2 + 16x_2^2$ is positive definite

c) Compute dV/dt: 6

$$\begin{aligned}
\dot{V} &= \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2 \\
&= 8x_1(x_1 x_2^2 - 9x_1 - 16x_2^3) + 32x_2(4x_1 x_2^2 + 2x_2 x_1^2) \\
&= 8x_1^2 x_2^2 - 72x_1^2 \underline{-128x_1 x_2^3} + \underline{128x_1 x_2^3} + 64x_2^2 x_1^2 \\
&= -8x_1^2[9 - x_2^2(1+8)] \\
&= -72x_1^2(1 - x_2^2)
\end{aligned}$$

$\Rightarrow \dot{V} = 0$ for $(0, x_2)$
$\Rightarrow \dot{V} \leq 0$ for $|x_2| < 1$
$\Rightarrow$ is negative semi-definite

$\Rightarrow V$ is a weak Lyapunov function
$\Rightarrow$ by the Lyapunov stability theorem follows that the origin is a stable fixed point. [3]

(*ii*) For $|x_2| < 1$ all points inside the level curve of $V(x_1, x_2) = 4x_1^2 + 16x_2^2$ will be dragged to the origin.

$$\Rightarrow 4x_1^2 + 16x_2^2 < 16 \quad \Rightarrow \frac{x_1^2}{4} + x_2^2 < 1$$

$\Rightarrow$ the length of the minor is 2 and the length of the major is 4. [4]

(*iii*) **Corollary:** *Let $V(\vec{x}(t))$ be a weak Lyapunov function for the system $\dot{\vec{x}} = \vec{X}(\vec{x})$ in a neighbourhood of an isolated fixed point $\vec{x}_f = (0,0)$. Then if $\dot{V} \neq 0$ on a trajectory, except for the fixed point, the origin is asymptotically stable.*
For $(0, x_2)$ the dynamical system reduces to $\dot{x}_1 = -16x_2^3$ and $\dot{x}_2 = 0$. [3]
This means the line $(0, x_2)$ is not a trajectory and therefore it follows from the corollary that the origin is asymptotically stable. [$\sum = 20$]

2. We have

$$\begin{aligned} \dot{x}_1 &= x_1 - x_2 - x_1^3 \\ \dot{x}_2 &= x_1 + x_2 - x_2^3 \end{aligned}$$

(*i*) With $x_1 = r\cos\vartheta$ and $x_2 = r\sin\vartheta$ we obtain

$$\begin{aligned} \dot{x}_1 &= \dot{r}\cos\vartheta - r\sin\vartheta\dot{\vartheta} = r\cos\vartheta - r\sin\vartheta - r^3\cos^3\vartheta && (1) \\ \dot{x}_2 &= \dot{r}\sin\vartheta + r\cos\vartheta\dot{\vartheta} = r\sin\vartheta + r\cos\vartheta - r^3\sin^3\vartheta && (2) \end{aligned}$$

computing: (1) $\times \cos\vartheta$ + (2) $\times \sin\vartheta$:

$$\begin{aligned} \dot{r} &= r\cos^2\vartheta + r\sin^2\vartheta - r^3(\cos^4\vartheta + \sin^4\vartheta) \\ &= r\left[1 - \frac{r^2}{4}(3 + \cos 4\vartheta)\right] \end{aligned}$$

computing: (1) $\times \sin\vartheta$ - (2) $\times \cos\vartheta$:

$$r\dot{\vartheta} = r - r^3\sin^3\vartheta\cos\vartheta + r^3\cos^3\vartheta\sin\vartheta$$

dividing by r: [4]

$$\begin{aligned} \dot{\vartheta} &= 1 - r^2\sin\vartheta\cos\vartheta(\sin^2\vartheta - \cos^2\vartheta) \\ &= 1 - \frac{r^2}{2}\sin 2\vartheta\cos 2\vartheta \\ &= 1 - \frac{r^2}{4}\sin 4\vartheta \end{aligned}$$

For a further fixed point to exist we need to solve

$$1 - \frac{r^2}{4}(3 + \cos 4\vartheta) = 0 \qquad \text{and} \qquad 1 - \frac{r^2}{4}\sin 4\vartheta = 0$$

Substiting $r^2 = -4/\sin 4\vartheta$ into the first equation yields

$$3 + \cos 4\vartheta + \sin 4\vartheta = 0,$$

which has no solution. Therefore we have no further fixed point. 2

(ii) **Poincaré-Bendixson theorem:** *Let φ_t be a flow for the system $\dot{\vec{x}} = \vec{X}(\vec{x})$ and let $\mathcal{D}$ be a closed, bounded and connected set $\mathcal{D} \in \mathbb{R}^2$, such that $\varphi_t(\mathcal{D}) \subset \mathcal{D}$ for all time. Furthermore $\mathcal{D}$ does not contain any fixed point. Then there exists at least one limit cycle in $\mathcal{D}$.* 3

for $r = 1/3$ we compute:

$$\dot{r} = \frac{1}{3}\left[1 - \frac{1}{36}(3 + \cos 4\vartheta)\right] > 0$$

for $r = 2$ we compute:

$$\dot{r} = 2\left[1 - (3 + \cos 4\vartheta)\right] < 0$$

$\Rightarrow$ trajectories which enter the region

$$\mathcal{D} = \left\{(r, \vartheta) : \frac{1}{3} \leq r \leq 2\right\}$$

do not leave it anymore.

$\Rightarrow$ Since there is no fixed point in $\mathcal{D}$, see (i), we can employ the Poincaré-Bendixson theorem to deduce that there is at least one limit cycle in $\mathcal{D}$. 3

(iii) **Bendixson's criterium:** *Let $\mathcal{D}$ be a simply connected region of the phase plane in which the function $\vec{X}(\vec{x})$ of the system $\dot{\vec{x}} = \vec{X}(\vec{x})$ has the property that its divergence is of constant sign, i.e.*

$$\operatorname{div} \vec{X} = \frac{\partial X_1}{\partial x_1} + \frac{\partial X_2}{\partial x_2} < 0 \qquad \text{or} \qquad \operatorname{div} \vec{X} = \frac{\partial X_1}{\partial x_1} + \frac{\partial X_2}{\partial x_2} > 0.$$

Then the system has no closed orbit contained entirely in $\mathcal{D}$. 2

We also have the theorem: *A limit cycle contains at least one fixed point.* 1

For the system

$$\begin{aligned} \dot{x}_1 &= x_1^2 - x_2 - 1 \\ \dot{x}_2 &= x_1 x_2 - 2x_2 \end{aligned}$$

we compute

$$\operatorname{div} \vec{X} = \frac{\partial X_1}{\partial x_1} + \frac{\partial X_2}{\partial x_2} = 3x_1 - 2 = \begin{cases} < 0 & \text{for } x_1 < 2/3 \\ = 0 & \text{for } x_1 = 2/3 \\ > 0 & \text{for } x_1 > 2/3 \end{cases}$$

This means if a limit cycle exists it has to cross the line $x_1 = 2/3$.
Find the fixed points from

$$x_1^2 - x_2 - 1 = 0 \quad \text{and} \quad x_1 x_2 - 2x_2 = 0.$$

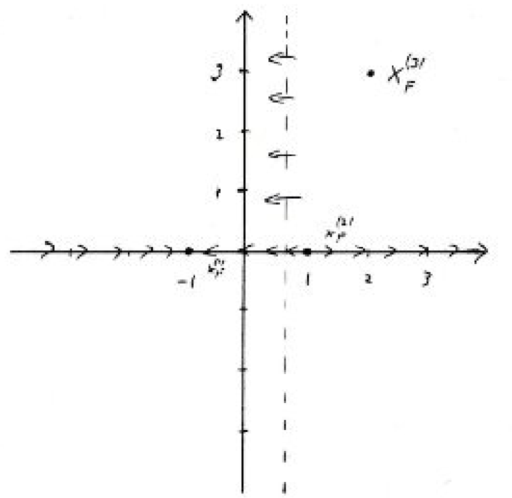
This yields

$$\vec{x}_f^{(1)} = (-1, 0), \qquad \vec{x}_f^{(2)} = (1, 0), \qquad \vec{x}_f^{(3)} = (2, 3).$$

We also have

$$\begin{aligned} \dot{x}_1 &= -\frac{5}{9} - x_2, & \dot{x}_2 &= \frac{2}{3}x_2 - 2x_2 & &\text{for } x_1 = 2/3, \\ \dot{x}_1 &= x_1^2 - 1, & \dot{x}_2 &= 0 & &\text{for } x_2 = 0. \end{aligned}$$

This means we can never encircle a fixed point and cross the $x_1 = 2/3$ at the same time. We assemble these data into the following figure 5



$\sum = 20$

3. (i) In one dimension *Bifurcation theory* investigates how the number of steady solutions of the system $\dot{x} = F(x, \lambda)$ depend on the parameter λ. A *bifurcation* occurs if the solution of $\dot{x} = F(x, \lambda)$ changes its qualitative behaviour as λ varies.

Def.: *Let (x_0, λ_0) be a fixed point for the system $\dot{x} = F(x, \lambda)$. If $\partial F/\partial\lambda|_{(x_0,\lambda_0)} \neq 0$ and $\partial\lambda/\partial x$ changes sign at (x_0, λ_0), then (x_0, λ_0) is called a turning point.*

Def.: *Let (x_0, λ_0) be a fixed point for the system $\dot{x} = F(x, \lambda)$. If $\partial F/\partial\lambda|_{(x_0,\lambda_0)} = 0$ and $\partial F/\partial x|_{(x_0,\lambda_0)} = 0$ and if through (x_0, λ_0) pass two and only two braches of the equilibrium curve which have both distinct tangents at (x_0, λ_0), then (x_0, λ_0) is called a transcritical bifurcation.*

Def.: *Let (x_0, λ_0) be a fixed point for the system $\dot{x} = F(x, \lambda)$.If $\partial F/\partial\lambda|_{(x_0,\lambda_0)} = 0$ and $\partial F/\partial x|_{(x_0,\lambda_0)} = 0$ and $d\lambda/dx$ changes sign on one branch of the equilibrium curve with distinct tangents, then (x_0, λ_0) is called a pitchfork bifurcation.* 4

(ii) We have the following **corollary:** *Suppose that for the system $\dot{\vec{x}} = \vec{X}(\vec{x})$we have transformed the linearized system $\dot{\vec{x}} = A\vec{x}$ with the help of $\vec{x} = U\vec{y}$ into the Jordan normal form*

$$\dot{\vec{y}} = U^{-1}AU\vec{y} = J\vec{y} = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}, \tag{3}$$

where $\vec{x} = U\vec{y}$, $\dot{\vec{y}} = \vec{Y}(\vec{y})$. Then if the stability index

$$I = \omega\left(Y^1_{111} + Y^1_{122} + Y^2_{112} + Y^2_{222}\right) + Y^1_{11}(Y^2_{11} - Y^1_{12}) + Y^2_{22}(Y^2_{12} - Y^1_{22}) + Y^2_{11}Y^2_{12} - Y^1_{22}Y^1_{12}$$

computed from is negative, the origin is asymptotically stable.

We compute the Jacobian for the system

$$\dot{x}_1 = 7x_2 \qquad \dot{x}_2 = -(x_1^2 - \lambda)x_2 - 7x_1 - 2x_1^3$$

to

$$A = \begin{pmatrix} 0 & 7 \\ -7 & 0 \end{pmatrix}.$$

Note this is already in Jordan normal form, such that $A = J$ and $\vec{X} = \vec{Y}$. Therefore $\omega = 7$. The only nonvanishing term in I is $Y^2_{112} = -2$. This means

$$I = \omega Y^2_{112} = -14.$$

As I is negative it follows that the origin is asymptotically stable. 3

(iii) **Hopf bifurcation theorem:** *Let (0,0,λ) with $\lambda \in \mathbb{R}$ be a fixed point of the system*

$$\begin{aligned} \dot{x}_1 &= F(x_1, x_2, \lambda) \\ \dot{x}_2 &= G(x_1, x_2, \lambda). \end{aligned}$$

If

i) The eigenvalues $e_1(\lambda)$ and $e_2(\lambda)$ of the linearized system are purely imaginary for some value $\lambda = \tilde{\lambda}$, i.e. $e_1(\lambda) \in i\mathbb{R}$ and $e_2(\lambda) \in i\mathbb{R}$.

ii) The real part of the eigenvalues $\mathrm{Re}(e_{1/2}(\lambda))$ satisfies

$$\frac{d}{d\lambda}\mathrm{Re}(e_{1/2}(\lambda))\Big|_{\lambda=\tilde{\lambda}} > 0.$$

iii) The origin is asymptotically stable for $\lambda = \tilde{\lambda}$.

then

a) $\lambda = \tilde{\lambda}$ is a bifurcation point of the system.

b) For $\lambda \in (\lambda_1, \tilde{\lambda})$ with some $\lambda_1 < \tilde{\lambda}$ the origin is a stable focus.

c) For $\lambda \in (\tilde{\lambda}, \lambda_2)$ with some $\lambda_2 > \tilde{\lambda}$ the origin is an unstable focus surrounded by a stable limit cycle whose size increases with λ. State the Hopf bifurcation theorem and use it to prove that the system possesses a Hopf bifurcation for $\lambda = 0$. [3]

The Jacobian matrix for $\lambda \neq 0$ is

$$A = \begin{pmatrix} 0 & 7 \\ -7 & \lambda \end{pmatrix},$$

with eigenvalues $e_\pm = \lambda/2 \pm \sqrt{\lambda^2 - 196}$.

i) for $\lambda = 0$ the eigenvales are purely imaginary $e_\pm = \pm i7$.

ii) we compute

$$\frac{d}{d\lambda} \mathrm{Re}(e_{1/2}(\lambda))\bigg|_{\lambda=\tilde{\lambda}=0} = \frac{1}{2} > 0.$$

iii) from part (ii) of the question we know that the origin is asymptotically stable.

Therefore the Hopf bifurcation theorem applies. [2]

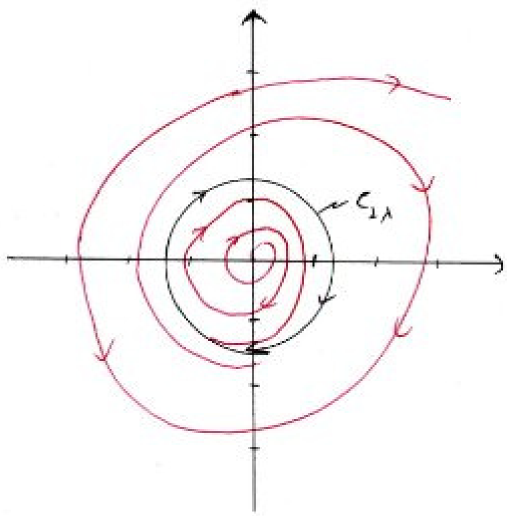
(iv) We have

$$\dot{r} = 0 \quad \text{for } r = 0, 2\lambda \qquad \text{and} \qquad \dot{r} > 0 \quad \text{for } r \neq 0, 2\lambda$$

Therefore

$$L_\alpha(\vec{x}) = \begin{cases} 0 & \text{for } 0 \leq r < \lambda \\ C_{2\lambda} & \text{for } r \neq \lambda \end{cases} \qquad L_\omega(\vec{x}) = \begin{cases} 0 & \text{for } r = 0 \\ C_{2\lambda} & \text{for } 0 < r \leq \lambda \\ \varnothing & \text{for } r > \lambda \end{cases}$$

Phase portrait:

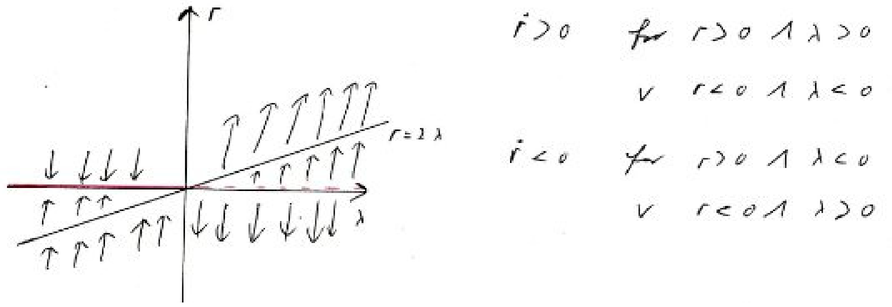


[4]

The fixed points are at $r = 0$ and $r = 2\lambda$. With $F(r, \lambda)$ follows

$$\frac{\partial F}{\partial r} = 4\lambda^3 - 8r\lambda^2 + 3r^2\lambda = 0 \quad \Rightarrow \lambda = 0$$
$$\frac{\partial F}{\partial \lambda} = r^3 - 8r^2\lambda + 12r\lambda^2 = 0 \quad \Rightarrow r = 0$$

which means there is a transcritical bifurcation point at $(0,0)$.
Bifurcation diagram: 4



$\sum = 20$

4. (i) **Def.:** *A system of differential equations on* $\mathbb{R}^2$ *is said to be a* <u>*Hamiltonian system*</u> *with one degree of freedom if there exists a twice continuously differentiable function* $H(x_1, x_2)$ *such that* 1

$$\dot{x}_1 = \frac{\partial H}{\partial x_2} \qquad \text{and} \qquad \dot{x}_2 = -\frac{\partial H}{\partial x_1}. \tag{4}$$

(ii) *Proof:* We compute the Jacobian matrix for the sytem (4) as

$$A = \begin{pmatrix} \frac{\partial^2 H}{\partial x_1 \partial x_2} & \frac{\partial^2 H}{\partial x_2^2} \\ -\frac{\partial^2 H}{\partial x_1^2} & -\frac{\partial^2 H}{\partial x_1 \partial x_2} \end{pmatrix}\Bigg|_{\vec{x}_f} =: \begin{pmatrix} H_{12} & H_{22} \\ -H_{11} & -H_{12} \end{pmatrix}\Bigg|_{\vec{x}_f}.$$

The eigenvalues are then obtained from

$$\det(A - \lambda \mathbb{I}) = (H_{12} - \lambda)(-H_{12} - \lambda) + H_{11}H_{22} = 0,$$

such that

$$\lambda^2 = -H_{11}H_{22} + H_{12}^2.$$

When the fixed point is nondegenerate we only have the two possibilities

$$H_{12}^2 - H_{11}H_{22} \begin{cases} > 0 & \equiv \text{ real eigenvalues of opposite sign } \equiv \text{ saddle point} \\ < 0 & \equiv \text{ purely complex eigenvalues } \equiv \text{ centre} \end{cases},$$

which is what we wanted to prove. 4

(iii) **Def.:** *A Hamiltonian system which is of the form*

$$H(x_1, x_2) = \frac{1}{2}x_2^2 + V(x_1),$$

where $V(x_1)$ *is a function which only depends on* x_1 *and not* x_2 *is called a* <u>*potential system*</u> *with* <u>*potential (function)*</u> $V(x_1)$.
The corresponding equations of motion are 1

$$\dot{x}_1 = \frac{\partial H}{\partial x_2} = x_2 \qquad \text{and} \qquad \dot{x}_2 = -\frac{\partial H}{\partial x_1} = -\frac{\partial V}{\partial x_1}.$$

For the system under consideration we have

$$\begin{aligned} V(x_1) &= \kappa e^{-x_1} \sin x_1, \\ V'(x_1) &= \kappa e^{-x_1}(\cos x_1 - \sin x_1), \\ V''(x_1) &= -2\kappa e^{-x_1} \cos x_1 \end{aligned}$$

The stationary points from $V'(x_1)$ are at

$$x_1^{(n)} = \frac{\pi}{4} + n\pi \qquad n \in \mathbb{N}.$$

Therefore 4

$$V''(x_1^{(n)}) = -2\kappa e^{-(\frac{\pi}{4}+n\pi)} \cos(\frac{\pi}{4} + n\pi) = (-1)^{n+1}\sqrt{2}\kappa e^{-(\frac{\pi}{4}+n\pi)}$$

for $\kappa \in \mathbb{R}^+, n$ even: $V''(x_1^{(n)}) < 0 \Rightarrow$maximum at $x_1^{(n)} \Rightarrow$saddle point at $(x_1^{(n)}, 0)$,
for $\kappa \in \mathbb{R}^+, n$ odd: $V''(x_1^{(n)}) > 0 \Rightarrow$minimum at $x_1^{(n)} \Rightarrow$centre at $(x_1^{(n)}, 0)$,
for $\kappa \in \mathbb{R}^-, n$ even: $V''(x_1^{(n)}) > 0 \Rightarrow$minimum at $x_1^{(n)} \Rightarrow$centre at $(x_1^{(n)}, 0)$,
for $\kappa \in \mathbb{R}^-, n$ odd: $V''(x_1^{(n)}) < 0 \Rightarrow$maximum at $x_1^{(n)} \Rightarrow$saddle point at $(x_1^{(n)}, 0)$. 2

(*iv*) For a potential system we have

$$H(x_1, x_2) = \frac{1}{2}x_2^2 + V(x_1),$$

Since $H(x_1, x_2)$ is conserved along a trajectory, i.e. $H(x_1, x_2) = E$ =const, we can write

$$E = \frac{1}{2}x_2^2 + V(x_1) \qquad \Rightarrow x_2 = \pm\sqrt{2[E - V(x_1)]}$$

For the period T we have to integrate along a trajectory

$$T = \oint_C dt = \oint_C dx_1/\dot{x}_1 = \oint_C dx_1/x_2 = \int_\alpha^\beta \frac{dx}{\sqrt{2[E - V(x)]}} + \int_\beta^\alpha \frac{dx}{-\sqrt{2[E - V(x)]}}$$

where α, β are the turning point, i.e. the values for x_1 when $x_2 = 0$. Therefore

$$T = 2\int_\alpha^\beta \frac{dx}{\sqrt{2[E - V(x)]}}$$

4

(*v*) First compute the constant E from

$$H(2^{3/4}, 2) = \frac{1}{2}2^2 + \frac{1}{4}2^3 = 4 = E.$$

The turning points result from solving

$$E = 4 = H(x_t, 0) = \frac{1}{4}x_t^4 \qquad \Rightarrow x_t^{(1/2)} = \pm 2.$$

Therefore

$$\begin{aligned} T &= 2\int_{-2}^{2} \frac{dx}{\sqrt{2[4-x^4/4]}} = 4\int_{0}^{2} \frac{dx}{\sqrt{2[4-x^4/4]}} \\ &= 4\int_{0}^{1} \frac{2dx}{\sqrt{8[1-x^4]}} = \sqrt{8}\int_{0}^{1} \frac{dx}{\sqrt{1-x^4}} \\ &= \sqrt{8\pi}\Gamma(5/4)\Gamma(3/4) \end{aligned}$$

4

5. (*i*) We can write $\boxed{\sum = 20}$

$$x_{n+1} = F(x_n) = (x_n - 3\lambda)(x_n + 5\lambda)$$

This means we have fixed points at 1

$$x_f^{(1)} = 3\lambda \qquad \text{and} \qquad x_f^{(2)} = -5\lambda.$$

A fixed point x_f is stable iff $|F'(x_f)| < 1$. With $F'(x) = 2x + 2\lambda$ follows that $x_f^{(1)}$ is stable for $|6\lambda + 2\lambda| < 1$, that is $\lambda < 1/8$.
$x_f^{(2)}$ is stable for $|-10\lambda + 2\lambda| < 1$, that is $\lambda < 1/8$. 3

(*ii*) A 2-cycle exists if $F(F(x)) = x$. Compute 1

$$\begin{aligned} x &= (F(x) - 3\lambda)(F(x) + 5\lambda) \\ &= F^2(x) + 2\lambda F(x) - 15\lambda \\ &= (x-3\lambda)^2(x+5\lambda)^2 + 2\lambda(x-3\lambda)(x+5\lambda) - 15\lambda \\ &= x^4 + 225\lambda^4 - 60x\lambda^3 + 4x^3\lambda - 26x^2\lambda^2 + 2\lambda(x-3\lambda)(x+5\lambda) - 15\lambda \\ &= x^4 - 15\lambda - 30\lambda^3 + 225\lambda^4 + 4x\lambda^2 + 2x^2\lambda - 60x\lambda^3 + 4x^3\lambda - 26x^2\lambda^2 \end{aligned}$$

Since the fixed point is a solution of this equation, we can factor our the term $F(x) - x$. Verify that: 5 4

$$\begin{aligned} &(F(x) - x)(1 + x + x^2 + 2\lambda + 2x\lambda - 15\lambda^2) \\ &= x^4 - 15\lambda - 30\lambda^3 + 225\lambda^4 + 4x\lambda^2 + 2x^2\lambda - 60x\lambda^3 + 4x^3\lambda - 26x^2\lambda^2 - x = 0 \end{aligned}$$

This means

$$1 + x + x^2 + 2\lambda + 2x\lambda - 15\lambda^2 = 0$$

for a two cycle to exist. Solving this quadratic equation gives

$$x_\pm = -(\lambda + \frac{1}{2}) \pm \frac{1}{2}\sqrt{64\lambda^2 - 4\lambda - 3}$$

For this to be real we require

$$64\lambda^2 - 4\lambda - 3 \geq 0.$$

Therefore the existence of a two cycle is ensured iff

$$(\lambda - \frac{1}{4})(\lambda + \frac{3}{16}) \geq 0,$$

which means $\lambda \geq 1/4$. 3

(iii) The 2 cycle is stable of for $G(x) = F(F(x))$

$$\left|G'(x)\right| < 1 \qquad \Leftrightarrow \qquad \left|F'(x_+)F'(x_-)\right| < 1$$

Compute

$$|(2x_+ + 2\lambda)(2x_- + 2\lambda)| = \left|-64\lambda^2 + 4\lambda + 4\right| < 1$$

This means

$$-64\lambda^2 + 4\lambda + 4 - 1 = (\lambda - \frac{1}{4})(\lambda + \frac{3}{16}) < 0 \qquad \Rightarrow \lambda < \frac{1}{4}$$
$$-64\lambda^2 + 4\lambda + 4 + 1 = (\lambda + \frac{1}{4})(\lambda - \frac{5}{16}) > 0 \qquad \Rightarrow \lambda > \frac{5}{16}$$

The domain of stability for the 2 cycle is empty. The two cycle is always unstable. $\boxed{3}$

$\boxed{\sum = 20}$