MA3608 Dynamical Systems II

Solutions and marking scheme for exam January 2010

INSTRUCTIONS: Full marks are obtained for correct answers to three of the five questions. Each question carries 20 marks.
(All questions are unseen, apart from definitions and theorems which are seen.)

1. (i) Defining $\boxed{2}$

$$x_1 = x \qquad \text{and} \qquad x_2 = \dot{x},$$

we obtain

$$\begin{aligned} \dot{x}_1 &= \dot{x} = x_2, \\ \dot{x}_2 &= \ddot{x} = -x_2 - \mu x_1^3 - \nu x_2^5. \end{aligned}$$

(ii) The fixed point $\vec{x}_f = (0,0)$ results from $\boxed{1}$

$$\begin{aligned} x_2 &= 0, \\ -x_2 - \mu x_1^3 - \nu x_2^5 &= 0. \end{aligned}$$

Linearization theorem: *Consider a nonlinear system which possesses a simple linearization at some fixed point. Then in a neighbourhood of the fixed point the phase portraits of the linear system and its linearization are qualitatively equivalent, if the eigenvalues of the Jacobian matrix have a nonzero real part, i.e. the linearized system is not a centre.* $\boxed{2}$

We compute the Jacobian for the above system at the fixed point:

$$A(\vec{x}_f) = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \Rightarrow \det A(\vec{x}_f) = 0 \Rightarrow \text{non-simple linearization}$$

The linearization theorem can not be applied since the system is non-simple.

(iii) **Lyapunov stability theorem:** *Consider the system $\dot{\vec{x}} = \vec{F}(\vec{x})$ with a fixed point at the origin. If there exists a real valued function $V(\vec{x})$ in a neighbourhood $N(\vec{x} = 0)$ such that:* $\boxed{3}$

i) *the partial derivatives $\partial V/\partial x_1$, $\partial V/\partial x_2$ exist and are continuous,*

ii) *the function $V(\vec{x})$ is positive definite,*

iii) *dV/dt is negative semi-definite (definite),*

then the origin is a stable (asymptotically stable) fixed point.

Verify conditions: $\boxed{5}$

i) the partial derivatives $\partial V/\partial x_1 = 4\alpha x_1^3$, $\partial V/\partial x_2 = 4x_2$ exist and are continuous,

ii) the function $V(\vec{x})$ is positive definite, i.e. $V(\vec{0}) = 0$ and $V(\vec{x}) > 0 \forall \vec{x} \neq \vec{0}$,

iii) dV/dt should be negative semi-definite for $V(\vec{x})$ to be a weak Lyapunov function:

$$\begin{aligned}\frac{dV}{dt} &= \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2 \\ &= 4\alpha x_1^3 x_2 + 4x_2(-x_2 - \mu x_1^3 - \nu x_2^5) \\ &= (4\alpha - 4\mu)x_1^3 x_2 - 4x_2^2 - 4\nu x_2^6\end{aligned}$$

$\Rightarrow$ dV/dt should be negative semi-definite for $\alpha = \mu$ and $\nu \geq 0$.

(*iv*) The partial derivatives $\partial V_1/\partial x_1$ and $\partial V_1/\partial x_2$ exist and are continuous. $V_1(\vec{x})$ is positive definite. But $\boxed{3}$

$$\begin{aligned}\frac{dV}{dt} &= \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2 \\ &= 2x_1x_2 + 2x_2(-x_2 - x_1^3)\end{aligned}$$

does not lead to a negative semi-definite function. Therefore V_1 is not a Lyapunov function.

$V_2(\vec{x})$ is not positive definite. Therefore V_2 is not a Lyapunov function.

(*v*) **Corollary:** *Let $V[\vec{x}(t)]$ be a weak Lyapunov function for the system $\dot{\vec{x}} = \vec{F}(\vec{x})$ in a neighbourhood of the isolated fixed point $\vec{x}_f = (0,0)$. Then if $\dot{V} \neq 0$ on any trajectory, except for the fixed point, the origin is asymptotically stable.* $\boxed{4}$

- We have $dV/dt = 0$ for $\vec{x} = (x_1, 0)$.
- On this line we have $\dot{x}_1 = 0$ and $\dot{x}_2 = -\mu x_1^3$, which means the line $\vec{x} = (x_1, 0)$ is not a trajectory.
- Therefore $\vec{x} = (0,0)$ is asymptotically stable. $\boxed{\sum = 20}$

2. We have

$$\begin{aligned}\dot{x}_1 &= x_2 + x_1(x_1^2 + x_2^2 - 5)(1 - x_1^4 - x_2^4 - 2x_1^2x_2^2) \\ \dot{x}_2 &= -x_1 + x_2(x_1^2 + x_2^2 - 5)(1 - x_1^4 - x_2^4 - 2x_1^2x_2^2)\end{aligned}$$

(*i*) With $x_1 = r\cos\vartheta$ and $x_2 = r\sin\vartheta$ we obtain $\boxed{4}$

$$\dot{x}_1 = \dot{r}\cos\vartheta - r\sin\vartheta\dot{\vartheta} = r\sin\vartheta + r\cos\vartheta(r^2 - 5)(1 - r^4) \quad (1)$$

$$\dot{x}_2 = \dot{r}\sin\vartheta + r\cos\vartheta\dot{\vartheta} = -r\cos\vartheta + r\sin\vartheta(r^2 - 5)(1 - r^4) \quad (2)$$

where $x_1^2 + x_2^2 = r^2$. Computing (1) $\times \cos\vartheta$ + (2) $\times$ $\sin\vartheta$ gives

$$\dot{r} = r(r^2 - 5)(1 - r^4). \tag{3}$$

Next we compute (1) $\times \sin\vartheta$ - (2) $\times$ $\cos\vartheta$:

$$-r\dot{\vartheta} = r$$

Dividing by r gives

$$\dot{\vartheta} = -1. \tag{4}$$

Since $\dot{\vartheta} \neq 0$ the origin is the only fixed point. $\boxed{1}$

(*ii*) **Poincaré-Bendixson theorem:** *Let φ_t be a flow for the system $\dot{\vec{x}} = \vec{F}(\vec{x})$ and let $\mathcal{D}$ be a closed, bounded and connected set $\mathcal{D} \in \mathbb{R}^2$, such that $\varphi_t(\mathcal{D}) \subset \mathcal{D}$ for all time. Furthermore $\mathcal{D}$ does not contain any fixed point. Then there exists at least one limit cycle in $\mathcal{D}$.* $\boxed{2}$

For $r = 2$ we compute $\boxed{3}$

$$\dot{r}(2) = 2(4-5)(1-16) = 30 > 0,$$

and for $r = 3$ we compute

$$\dot{r}(3) = 2(9-5)(1-81) = -960 < 0.$$

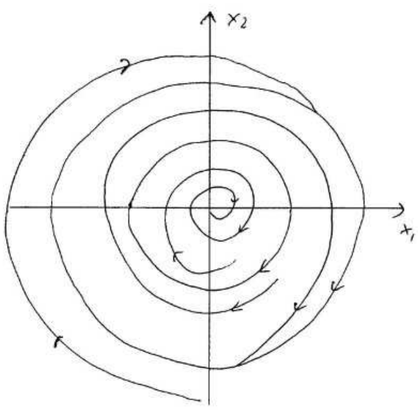
$\Rightarrow$ trajectories which enter the region

$$\mathcal{D} = \{(r, \vartheta) : 2 \leq r \leq 3\}$$

can never leave it.

$\Rightarrow$ Since there is no fixed point in $\mathcal{D}$, see (*i*), we can employ the Poincaré-Bendixson theorem to deduce that there is at least one limit cycle in $\mathcal{D}$.

(*iii*) Equations (3) and (4) give the diagram: $\boxed{2}$



$\dot{\vartheta} = -1$

$\dot{r} = (r^2-5)(1-r^4)r$

$\dot{r} < 0 \quad 0 < r < 1$

$\dot{r} > 0 \quad 1 < r < \sqrt{5}$

$\dot{r} < 0 \quad \sqrt{5} < r$

(*iv*) **Def.:** *The ω-limit set (or positive limit set) $L_\omega(\vec{x})$ of a point $\vec{x}$ contains those points which are approached by the trajectory through $\vec{x}$ as $t \to \infty$, that is* $\boxed{1}$

$$L_\omega(\vec{x}) = \Big\{\vec{y} \in \mathbb{R}^n : \exists \text{ a sequence of times } t_n \text{ with } t_n \to \infty, \text{ such that } \lim_{n\to\infty} \varphi_{t_n}(\vec{x}) = \vec{y}\Big\}$$

Def.: *The α-limit set (or negative limit set) $L_\alpha(\vec{x})$ of a point $\vec{x}$ contains those points which are approached by the trajectory through $\vec{x}$ as $t \to -\infty$, that is*

$$L_\alpha(\vec{x}) = \Big\{\vec{y} \in \mathbb{R}^n : \exists \text{ a sequence of times } t_n \text{ with } t_n \to -\infty, \text{ such that } \lim_{n\to\infty} \varphi_{t_n}(\vec{x}) = \vec{y}\Big\}$$

Accordingly we compute for (3) and (4) $\boxed{3}$

$$L_\alpha(\vec{x}) = \begin{cases} 0 & \text{for } r = 0 \\ \mathcal{C}_1 & \text{for } 0 < r < \sqrt{5} \\ \mathcal{C}_{\sqrt{5}} & \text{for } r = \sqrt{5} \\ \varnothing & \sqrt{5} < r \end{cases} \qquad L_\omega(\vec{x}) = \begin{cases} 0 & \text{for } 0 < r < 1 \\ \mathcal{C}_1 & \text{for } r = 1 \\ \mathcal{C}_{\sqrt{5}} & \text{for } 1 < r \end{cases}$$

Def.: *A closed orbit ϕ is a limit cycle if ϕ is a subset of an α or ω-limit set for some point $\vec{x} \notin \phi$.* $\boxed{2}$
We have $\dot{r} = 0$ for $r = 0, \sqrt{5}, 1$, which means we have a limit cycle with radius $r = 1 : \mathcal{C}_1$ and one with radius $r = \sqrt{5} : \mathcal{C}_{\sqrt{5}}$.
Def.: *A limit cycle ϕ is a called a stable (unstable) limit cycle, if $\phi = L_\omega(\vec{x})$ ($\phi = L_\alpha(\vec{x})$) for all $\vec{x}$ in some neighbourhood of the limit cycle.*
Def.: *A limit cycle ϕ is a called a semi-stable limit cycle, if it is a stable limit cycle for points on one side and an unstable limit cycle for point on the other side.*

Therefore $\mathcal{C}_1$ is unstable and $\mathcal{C}_{\sqrt{5}}$ is stable. The limit cycle $\mathcal{C}_{\sqrt{5}}$ is the one identified in (*ii*) since $\mathcal{C}_{\sqrt{5}} \subset \mathcal{D}$.

(*v*) **Bendixson's criterium:** *Let $\mathcal{D}$ be a simply connected region of the phase plane in which the function $\vec{F}(\vec{x})$ of the system $\dot{\vec{x}} = \vec{F}(\vec{x})$ has the property that its divergence is of constant sign, i.e.* $\boxed{2}$

$$\operatorname{div} \vec{F} = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} < 0 \qquad \text{or} \qquad \operatorname{div} \vec{F} = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} > 0.$$

Then the system posesses no closed orbit contained entirely in $\mathcal{D}$.

$\hat{\mathcal{D}}$ is not a simply connected region and therefore we can not apply Bendixson's criterium to decide whether it contains limit cycles or not. $\boxed{\sum = 20}$

3. (*i*) *Bifurcation theory* investigates how the number of steady solutions of systems of the type $\dot{x} = F(x, \lambda)$ depend on the parameter λ. A *bifurcation* occurs if the solution of $\dot{x} = F(x, \lambda)$ changes its qualitative behaviour as the parameter $\boxed{2}$

λ varies. Considering $F(x, \lambda) = 0$ leads to a plot in the (x, λ)-plane called the *bifurcation diagram.*

The fixed points are found from $\boxed{2}$

$$F(x, \lambda) = x^3 + \gamma x^2 - \lambda x = 0.$$

i.e. they are at the three curves

$$x_1 = 0 \qquad x_{2/3} = \frac{1}{2}\left(-\gamma \pm \sqrt{\gamma^2 + 4\lambda}\right).$$

In order to characterize the types of bifurcations we need

$$\frac{\partial F(x, \lambda)}{\partial x} = 3x^2 + 2\gamma x - \lambda \qquad \text{and} \qquad \frac{\partial F(x, \lambda)}{\partial \lambda} = -x.$$

- *A point (x_0, λ_0) on the equilibrium curve for the system $\dot{x} = F(x, \lambda)$ is called a <u>pitchfork bifurcation</u> if $\partial F/\partial \lambda|_{(x_0,\lambda_0)} = 0$, $\partial F/\partial x|_{(x_0,\lambda_0)} = 0$ and $d\lambda/dx$ changes sign on one branch of the equilibrium curve with distinct tangents, where $\lambda(x)$ is the solution of the equation $F(x, \lambda) = 0$.* $\boxed{2}$
  $\partial F/\partial \lambda|_{(x_0,\lambda_0)} = 0$ gives $x_0 = 0$ and subsequently $\partial F/\partial x|_{(x_0,\lambda_0)} = 0$ gives $\lambda_0 = 0$. Since $d\lambda/dx = 2x$ changes sign at $x_0 = 0$ and this branch has a different tangent than $x_1 = 0$, the point $(x_0, \lambda_0) = (0, 0)$ constitutes a pitchfork bifurcation.
- *A point (x_0, λ_0) on the equilibrium curve for the system $\dot{x} = F(x, \lambda)$ is called a <u>transcritical bifurcation</u> if $\partial F/\partial \lambda|_{(x_0,\lambda_0)} = 0$, $\partial F/\partial x|_{(x_0,\lambda_0)} = 0$ and in addition two and only two branches of the equilibrium curve pass through this point which have both distinct tangents at (x_0, λ_0).* $\boxed{2}$
  For $\gamma \neq 0$ we have $d\lambda/dx = 2x + \gamma$, which no longer changes sign at $x_0 = 0$. However, only two branches pass through this point and their tangents are distinct, such that $(x_0, \lambda_0) = (0, 0)$ constitutes a transcritical bifurcation.
- *A point (x_0, λ_0) on the equilibrium curve for the system $\dot{x} = F(x, \lambda)$ is called a <u>turning point</u> if $\partial F/\partial \lambda|_{(x_0,\lambda_0)} \neq 0$ and $\partial \lambda/\partial x$ changes sign at this point.* $\boxed{4}$
  From
  $$\frac{\partial x_2}{\partial \lambda} = \frac{1}{\sqrt{\gamma^2 + 4\gamma}} \qquad \text{and} \qquad \frac{\partial x_3}{\partial \lambda} = -\frac{1}{\sqrt{\gamma^2 + 4\gamma}}.$$
  follows that $\partial x/\partial \lambda$ changes sign for $\lambda_0 = -\gamma^2/4$, such that $x_2(\lambda_0) = x_3(\lambda_0) = x_0 = -\gamma/2$. Since $\partial F/\partial \lambda|_{(x_0,\lambda_0)} = \gamma/2 \neq 0$ this mean
  $$(x_0, \lambda_0) = (-\gamma/2, -\gamma^2/4)$$
  is a turning point for the above system.

(ii) We make use of the follwoing corollary: *Suppose that for the system $\dot{\vec{x}} = \vec{F}(\vec{x})$,we have transformed the linearized system $\dot{\vec{x}} = A\vec{x}$,with the help of $\vec{x} = U\vec{y}$ into the Jordan normal form* $\boxed{4}$

$$\dot{\vec{y}} = U^{-1}AU\vec{y} = J\vec{y} = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}, \qquad (5)$$

with $\omega \in \mathbb{R}^+$. Then the origin is asymptotically stable if the stability index I, computed from the transformed system $\dot{\vec{y}} = \vec{Y}(\vec{y})$, is negative.

Thus we compute the Jacobian matrix for the system

$$\begin{aligned} \dot{x}_1 &= 9x_2 + 3x_1^2 \\ \dot{x}_2 &= \lambda x_2 - 2x_1^2 x_2 - 9x_1 - 2x_1^3 + \alpha x_1^2 \end{aligned}$$

to

$$A(\vec{x}_f) = \begin{pmatrix} 0 & 9 \\ -9 & \lambda \end{pmatrix}$$

We note that for $\lambda = 0$ this is already in Jordan normal form, such that $A = J$ and $\vec{X} = \vec{Y}$. Therefore $\omega = 9$. The only nonvanishing terms in I are

$$Y^2_{112} = -4, \quad Y^2_{11} = 2\alpha \quad \text{and} \quad Y^1_{11} = 6.$$

Therefore

$$I = \omega Y^2_{112} + Y^2_{11} Y^1_{11} = 9(-4) + 12\alpha = 12\alpha - 36.$$

This means I is negative for $\alpha < 3$, i.e. the origin is asymptotically stable for $\alpha < 3$.

(iii) **Hopf bifurcation theorem:** *Let the point (0,0,λ), with $\lambda \in \mathbb{R}$, be a fixed point for the system* $\boxed{4}$

$$\begin{aligned} \dot{x}_1 &= F_1(x_1, x_2, \lambda), && (6) \\ \dot{x}_2 &= F_2(x_1, x_2, \lambda), && (7) \end{aligned}$$

for all values of λ. If for a particular value of λ, say $\lambda = \tilde{\lambda}$,

i) *the eigenvalues $e_1(\lambda)$ and $e_2(\lambda)$ of the linearized system are purely imaginary, i.e. $e_1(\tilde{\lambda}) \in iR$ and $e_2(\tilde{\lambda}) \in iR$,*

ii) *the real part of the eigenvalues $\mathrm{Re}(e_1(\lambda)) = \mathrm{Re}(e_2(\lambda))$ satisfies*

$$\left. \frac{d}{d\lambda} \mathrm{Re}(e_{1/2}(\lambda)) \right|_{\lambda=\tilde{\lambda}} > 0, \tag{8}$$

iii) *the origin is asymptotically stable for $\lambda = \tilde{\lambda}$,*

then the following statements hold:

a) *The point with $\lambda = \tilde{\lambda}$ is a bifurcation point of the system.*

b) *For $\lambda \in (\lambda_1, \tilde{\lambda})$ with some $\lambda_1 < \tilde{\lambda}$ the origin is a stable focus.*

c) *For $\lambda \in (\tilde{\lambda}, \lambda_2)$ with some $\lambda_2 > \tilde{\lambda}$ the origin is an unstable focus surrounded by a stable limit cycle whose size increases with λ.*

The eigenvalues for the Jacobian matrix with $\lambda \neq 0$ are computed to

$$e_\pm = \frac{1}{2}\left(\lambda \pm \sqrt{\lambda^2 - 324}\right).$$

i) for $\lambda = 0$ the eigenvales are purely imaginary: $e_\pm = \pm i9$.

ii) we compute

$$\frac{d}{d\lambda}\operatorname{Re}(e_{1/2}(\lambda))\Big|_{\lambda=\tilde{\lambda}=0} = \frac{1}{2} > 0.$$

iii) from part (ii) of the question we know that the origin is asymptotically stable for $\alpha = 2$.
Therefore the Hopf bifurcation theorem applies for $\alpha = 2$. The situation is inconclusive for $\alpha = 4$. $\boxed{\sum = 20}$

4. (i) **Def.:** *A system of differential equations on* $\mathbb{R}^2$ *is said to be a* <u>*Hamiltonian system*</u> *with one degree of freedom if there exists a twice continuously differentiable function* $H(x_1, x_2)$ *such that* $\boxed{2}$

$$\dot{x}_1 = \frac{\partial H}{\partial x_2} \qquad \text{and} \qquad \dot{x}_2 = -\frac{\partial H}{\partial x_1}. \tag{9}$$

The equations (9) are said to be the *equations of motions* correponding to the Hamiltonian H. When H does not depend explicitly on the time t, i.e. it is of the form $H(x_1(t), x_2(t))$ and not $H(x_1(t), x_2(t), t)$, the system is called *autonomous.*

(ii) *A dynamical system* $\boxed{2}$

$$\dot{x}_1 = F_1(x_1, x_2) \qquad \text{and} \qquad \dot{x}_2 = F_2(x_1, x_2),$$

is a Hamiltonian system if and only if

$$\operatorname{div}\vec{F} = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} = 0.$$

We compute

$$\operatorname{div}\vec{F} = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} = 3\mu x_1^2 x_2^2 + 2 - 6x_1^2 x_2^2 - 2 = 0.$$

Therefore the system is a Hamiltonian system when $\mu = 2$.

(iii) **Def.:** *A Hamiltonian system which is of the form* $\boxed{5}$

$$H(x_1, x_2) = \frac{1}{2}x_2^2 + V(x_1),$$

where $V(x_1)$ *is a function which only depends on* x_1 *and not* x_2 *is called a* <u>*potential system*</u> *with* <u>*potential (function)*</u> $V(x_1)$*.*
From the definition in (i) follows

$$\begin{aligned}
\dot{x}_1 &= \frac{\partial H}{\partial x_2} = x_2 \Rightarrow H(x_1, x_2) = \frac{1}{2}x_2^2 + f(x_1) \\
\dot{x}_2 &= -\frac{\partial H}{\partial x_1} = -2x_1 + \frac{20x_1}{1 + x_1^2} \Rightarrow H(x_1, x_2) = x_1^2 - 10\ln(1 + x^2) + f(x_2).
\end{aligned}$$

Therefore

$$H(x_1, x_2) = \frac{1}{2}x_2^2 + x_1^2 - 10\ln(1 + x_1^2) + c,$$

such that the potential is

$$V(x_1) = x_1^2 - 10\ln(1 + x_1^2) + c.$$

From $V(0) = 0$ follows $c = 0$.

(*iv*) *The fixed points for the Hamiltonian system described by* $\boxed{5}$

$$H(x_1, x_2) = \frac{1}{2}x_2^2 + V(x_1) \tag{10}$$

are located at the points $(a_k, 0)$ *with* $k = 1, 2, 3, \ldots$*, where the* a_k *are stationary points of the potential* $V(x_1)$*. If* $V(a_k)$ *is a minimum then the point* $(a_k, 0)$ *is a centre and if on the other hand* $V(a_k)$ *is a maximum the point* $(a_k, 0)$ *is a saddle point.*

We compute the stationary points from

$$V'(x_1) = 2x_1 - \frac{20x_1}{1 + x_1^2} = \frac{2x_1(x_1^2 - 9)}{1 + x_1^2} = 0 \qquad \text{for } x_1 = 0, \pm 3.$$

Furthermore

$$V''(x_1) = 2 - 10\left(-\frac{4x_1^2}{(1 + x_1^2)^2} + \frac{2}{1 + x_1^2}\right)$$

and therefore

$$\begin{aligned} V''(0) &= -18 \Rightarrow x_1 = 0 \text{ is a minimum of } V(x_1) \Rightarrow (0,0) \text{ is a saddle point,} \\ V''(\pm 3) &= \frac{18}{5} \Rightarrow x_1 = \pm 3 \text{ are maxima of } V(x_1) \Rightarrow (\pm 3, 0) \text{ are centres.} \end{aligned}$$

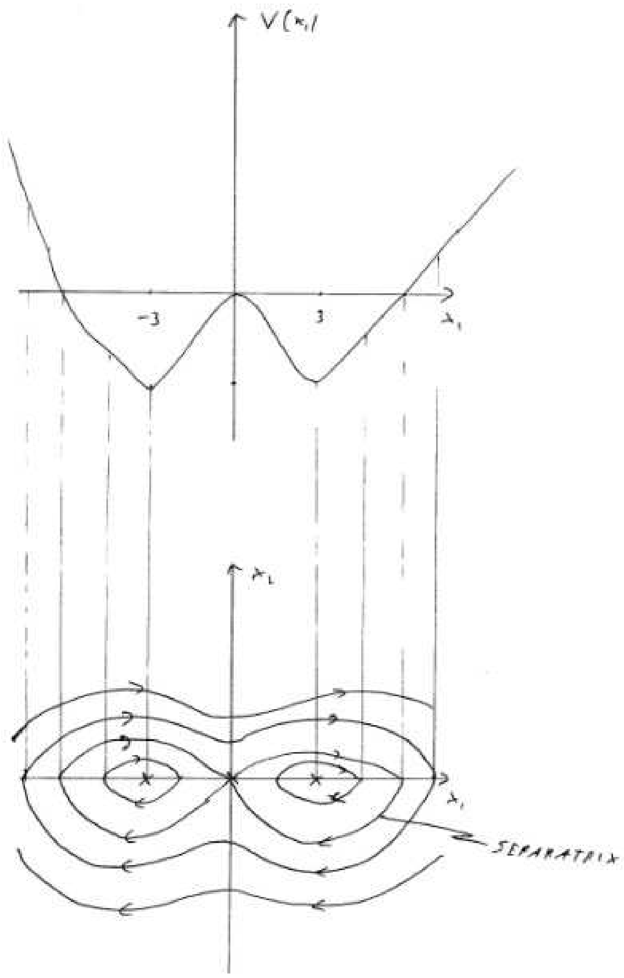
(*v*) The separatrix crosses the saddle point, i.e. $H(0, 0) = 0$ is conserved on the separatrix. The equation for the separatrix is therefore $\boxed{6}$

$$0 = \frac{1}{2}x_2^2 + x_1^2 - 10\ln(1 + x_1^2) \Rightarrow x_2 = \pm\sqrt{2x_1^2 - 20\ln(1 + x_1^2)}.$$

The direction of time follows from $\dot{x}_1 > 0$ for $x_2 > 0$ and $\dot{x}_1 < 0$ for $x_2 < 0$.

All trajectories are bounded.

We assemble all the information in the diagram:



$\boxed{\sum = 20}$

5. (*i*) The fixed points are found from $\boxed{1}$

$$F(x) = x \qquad \Leftrightarrow \qquad 8\lambda x - 4\lambda x^2 = x$$

This means we have fixed points at $\boxed{3}$

$$x_f^{(1)} = 0 \qquad \text{and} \qquad x_f^{(2)} = 2 - \frac{1}{4\lambda}.$$

A fixed point x_f is stable iff $|F'(x_f)| < 1$. With $F'(x) = 8\lambda - 8\lambda x$ follows that $x_f^{(1)}$ is stable for $|8\lambda| < 1$, that is $\lambda < 1/8$.
$x_f^{(2)}$ is stable for $|2 - 8\lambda| < 1$, that is $1/8 < \lambda < 3/8$.

(*ii*) A 2-cycle exists if $F(F(x)) = x$. Compute $\boxed{6}$

$$\begin{aligned} x &= 8\lambda F(x) - 4\lambda F^2(x) \\ &= 8\lambda(8\lambda x - 4\lambda x^2) - 4\lambda(8\lambda x - 4\lambda x^2)^2 \\ &= 64\lambda^2 x - 64\lambda^3 x^4 + 256\lambda^3 x^3 - 256\lambda^3 x^2 - 32\lambda^2 x^2 \\ &= 32(2\lambda^2 x - 2\lambda^3 x^4 + 8\lambda^3 x^3 - 8\lambda^3 x^2 - \lambda^2 x^2) \end{aligned}$$

Since the fixed point is a solution of this equation, we can factor out the term $F(x) - x$. Not knowing the answer the can be done by polynomial devision, but in this case it is sufficient to verify that: $\boxed{4}$

$$
\begin{aligned}
&(F(x) - x)\,(1 + 8\lambda - 4x\lambda - 32x\lambda^2 + 16x^2\lambda^2) \\
&= 32(2\lambda^2 x - 2\lambda^3 x^4 + 8\lambda^3 x^3 - 8\lambda^3 x^2 - \lambda^2 x^2) - x = 0
\end{aligned}
$$

This means we require $\boxed{3}$

$$1 + 8\lambda - 4x\lambda - 32x\lambda^2 + 16x^2\lambda^2 = 0$$

for a two cycle to exist. Solving this quadratic equation gives

$$x_\pm = 1 + \frac{1}{8\lambda} \pm \frac{1}{8\lambda}\sqrt{64\lambda^2 - 16\lambda - 3}$$

For this to be real we require

$$64\lambda^2 - 16\lambda - 3 \geq 0.$$

Therefore the existence of a two cycle is ensured iff

$$(8\lambda + 1)(8\lambda - 1) \geq 0,$$

which means $\lambda \geq 3/8$.

(iii) The 2 cycle is stable for $G(x) = F(F(x))$ $\boxed{3}$

$$\left|G'(x)\right| < 1 \qquad \Leftrightarrow \qquad \left|F'(x_+)F'(x_-)\right| < 1$$

Compute therefore

$$|(8\lambda - 8\lambda x_+)(8\lambda - 8\lambda x_-)| = \left|4 + 16\lambda - 64\lambda^2\right| < 1$$

This means the two cycle is stable in the regime

$$\frac{3}{8} < \lambda < \frac{1}{8}(1 + \sqrt{6})$$

and unstable for $\lambda > (1 + \sqrt{6})/8$ $\boxed{\sum = 20}$