MA3608 Dynamical Systems II

Solutions and marking scheme for exam January 2011

INSTRUCTIONS: Full marks are obtained for correct answers to three of the five questions. Each question carries 20 marks.
(All questions are unseen, apart from definitions and theorems which are seen.)

1. (*i*) **Definition:** *In case the map $\vec{F}$ in $\dot{\vec{x}} = \vec{F}(\vec{x})$ is linear in $x_1, x_2, \ldots, x_n$ the dynamical system is called a linear dynamical system.* $\boxed{2}$

This means the system acquires the simpler form

$$\dot{\vec{x}} = \vec{F}(\vec{x}) = A\vec{x} \tag{1}$$

with A being an $n \times n$-matrix with constant entries, i.e. A_{ij} =const for $1 \leq i, j \leq n$.

Definition: *A linear system is called simple, if A is non-singular, i.e.* $\det A \neq 0$ *and A has non-zero eigenvalues.* $\boxed{1}$

(*ii*) Taking the matrix A to be in the most general form with arbitrary constants entries $a, b, c, d \in \mathbb{R}$, equation (1) for the fixed point becomes $\boxed{7}$

$$\vec{F}(\vec{x}_f) = A\vec{x}_f = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0. \tag{2}$$

From this follows

$$\left.\begin{array}{l} ax_1 + bx_2 = 0 \Leftrightarrow x_1 = -b/ax_2 \\ cx_1 + dx_2 = 0 \Leftrightarrow x_1 = -d/cx_2 \end{array}\right\} \Rightarrow (\det A)\, x_2 = 0 \tag{3}$$

Since the determinant of A is non-vanishing we conclude from the last equality in (3) that $x_2 = 0$. A similar argument leads to $x_1 = 0$. As there are no further solutions to (2), the only fixed point of this linear system is the origin □.

(*iii*) First we need to show that J is of diagonal form. From the eigenvalue equation $\boxed{10}$

$$A\vec{v}_\pm = \lambda_\pm \vec{v}_\pm \qquad \text{with } \lambda_+ \neq \lambda_- \tag{4}$$

we construct a matrix U which consists of the eigenvectors of A as column vectors

$$U = (\vec{v}_+, \vec{v}_-). \tag{5}$$

Then we compute

$$AU = (A\vec{v}_+, A\vec{v}_-) = (\lambda_+\vec{v}_+, \lambda_-\vec{v}_-) = UJ. \tag{6}$$

Since $\lambda_+ \neq \lambda_-$ the two eigenvalues $\vec{v}_+$ and $\vec{v}_-$ are linearly independent, such that the matrix U is nonsingular, i.e. $\det U \neq 0$. Therefore the inverse U^{-1} exists, such that

$$J = U^{-1}AU = \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix}. \tag{7}$$

Next we consider the dynamical system produced by the matrix J

$$\begin{pmatrix} \dot{y}_+ \\ \dot{y}_- \end{pmatrix} = \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix} \begin{pmatrix} y_+ \\ y_- \end{pmatrix}, \tag{8}$$

which is evidently solved by

$$y_\pm = k_\pm e^{\lambda_\pm t}, \tag{9}$$

such that

$$y_- = k_-(y_+/k_+)^{\lambda_+/\lambda_-}. \tag{10}$$

Clearly if $\lambda_+ > \lambda_- > 0$ we obtain from this the phase portrait of an unstable node and when $\lambda_- < \lambda_+ < 0$ we obtain the phase portrait for a stable node. $\boxed{\sum = 20}$

2. We have

$$\begin{aligned} \dot{x}_1 &= x_2 + x_1(2 - x_1^6 - x_2^6 - 3x_1^4x_2^2 - 3x_1^2x_2^4)(x_1^2 + x_2^2 - 6) \\ \dot{x}_2 &= -x_1 + x_2(2 - x_1^6 - x_2^6 - 3x_1^4x_2^2 - 3x_1^2x_2^4)(x_1^2 + x_2^2 - 6) \end{aligned}$$

(*i*) With $x_1 = r\cos\vartheta$ and $x_2 = r\sin\vartheta$ we obtain $\boxed{4}$

$$\dot{x}_1 = \dot{r}\cos\vartheta - r\sin\vartheta\dot{\vartheta} = r\sin\vartheta + r\cos\vartheta(2 - r^6)(r^2 - 6) \tag{11}$$

$$\dot{x}_2 = \dot{r}\sin\vartheta + r\cos\vartheta\dot{\vartheta} = -r\cos\vartheta + r\sin\vartheta(2 - r^6)(r^2 - 6) \tag{12}$$

where $x_1^2 + x_2^2 = r^2$. Computing (11) $\times \cos\vartheta$ + (12) $\times$ $\sin\vartheta$ gives

$$\dot{r} = r(2 - r^6)(r^2 - 6). \tag{13}$$

Next we compute (11) $\times \sin\vartheta$ - (12) $\times$ $\cos\vartheta$:

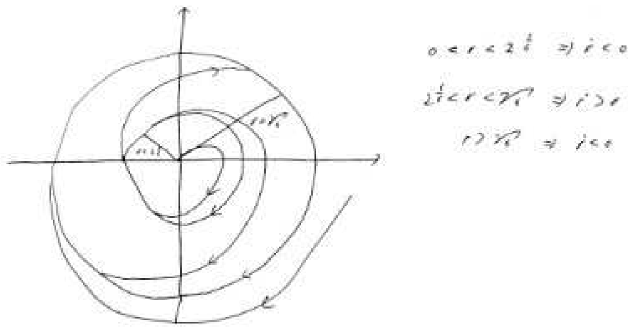
$$-r\dot{\vartheta} = r$$

Dividing by r gives

$$\dot{\vartheta} = -1. \tag{14}$$

Since $\dot{\vartheta} \neq 0$ the origin is the only fixed point. $\boxed{1}$

(*ii*) We have $\dot{r} = 0$ for $r = \sqrt{6}$ and $r = 2^{1/6}$, which means the limit cycles are at $r = \sqrt{6}\ ,2^{1/6}$. The equations (13) and (14) give the phase portrait: $\boxed{2}$



(*iii*) **Poincaré-Bendixson theorem:** *Let φ_t be a flow for the system $\dot{\vec{x}} = \vec{F}(\vec{x})$ and let $\mathcal{D}$ be a closed, bounded and connected set $\mathcal{D} \in \mathbb{R}^2$, such that $\varphi_t(\mathcal{D}) \subset \mathcal{D}$ for all time. Furthermore $\mathcal{D}$ does not contain any fixed point. Then there exists at least one limit cycle in $\mathcal{D}$.* $\boxed{2}$

For $r = 1$ we compute $\boxed{3}$

$$\dot{r}(1) = 1(2 - 1)(1 - 6) = -5 < 0,$$

and for $r = 3$ we have

$$\dot{r}(3) = 3(2 - 3^6)(3^2 - 6) = -6543 < 0.$$

$\Rightarrow$ trajectories which enter the region $\mathcal{D}$ can also leave it again. This means the Poincaré-Bendixson theorem can not be used to deduce that a limit cycle exists in $\mathcal{D}$.

(*iv*) **Def.:** *The <u>ω-limit set</u> (or positive limit set) $L_\omega(\vec{x})$ of a point $\vec{x}$ contains those points which are approached by the trajectory through $\vec{x}$ as $t \to \infty$, that is* $\boxed{1}$

$$L_\omega(\vec{x}) = \Big\{\vec{y} \in \mathbb{R}^n : \exists \text{ a sequence of times } t_n \text{ with } t_n \to \infty, \\ \text{such that } \lim_{n\to\infty} \varphi_{t_n}(\vec{x}) = \vec{y}\Big\}$$

Def.: *The <u>α-limit set</u> (or negative limit set) $L_\alpha(\vec{x})$ of a point $\vec{x}$ contains those points which are approached by the trajectory through $\vec{x}$ as $t \to -\infty$, that is*

$$L_\alpha(\vec{x}) = \Big\{\vec{y} \in \mathbb{R}^n : \exists \text{ a sequence of times } t_n \text{ with } t_n \to -\infty, \\ \text{such that } \lim_{n\to\infty} \varphi_{t_n}(\vec{x}) = \vec{y}\Big\}$$

Accordingly we compute for (13) and (14) $\boxed{3}$

$$L_\alpha(\vec{x}) = \begin{cases} 0 & \text{for } r = 0 \\ \mathcal{C}_{2^{1/6}} & \text{for } 0 < r < \sqrt{6} \\ \mathcal{C}_{\sqrt{6}} & \text{for } r = \sqrt{6} \\ \varnothing & r > \sqrt{6} \end{cases} \qquad L_\omega(\vec{x}) = \begin{cases} 0 & \text{for } 0 \le r < 1 \\ \mathcal{C}_{2^{1/6}} & \text{for } r = 2^{1/6} \\ \mathcal{C}_{\sqrt{6}} & \text{for } r > 2^{1/6} \end{cases}$$

Def.: *A closed orbit ϕ is a* <u>limit cycle</u> *if ϕ is a subset of an α or ω-limit set for some point $\vec{x} \notin \phi$.* $\boxed{2}$

Def.: *A limit cycle ϕ is a called a* <u>stable (unstable) limit cycle</u>*, if $\phi = L_\omega(\vec{x})$ $(\phi = L_\alpha(\vec{x}))$ for all $\vec{x}$ in some neighbourhood of the limit cycle.*

Def.: *A limit cycle ϕ is a called a* <u>semi-stable limit cycle</u>*, if it is a stable limit cycle for points on one side and an unstable limit cycle for point on the other side.*

Therefore $\mathcal{C}_{2^{1/6}}$ is unstable and $\mathcal{C}_{\sqrt{6}}$ is stable.

(v) **Bendixson's criterium:** *Let $\mathcal{D}$ be a simply connected region of the phase plane in which the function $\vec{F}(\vec{x})$ of the system $\dot{\vec{x}} = \vec{F}(\vec{x})$ has the property that its divergence is of constant sign, i.e.* $\boxed{2}$

$$\operatorname{div} \vec{F} = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} < 0 \qquad \text{or} \qquad \operatorname{div} \vec{F} = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} > 0.$$

Then the system posesses no closed orbit contained entirely in $\mathcal{D}$.

$\hat{\mathcal{D}}$ is not a simply connected region and therefore we can not apply Bendixson's criterium to decide whether it contains limit cycles or not. $\boxed{\sum = 20}$

3. (i) *Bifurcation theory* investigates how the number of steady solutions of systems of the type $\dot{x} = F(x, \lambda)$ depend on the parameter λ. A *bifurcation* occurs if the solution of $\dot{x} = F(x, \lambda)$ changes its qualitative behaviour as the parameter λ varies. $\boxed{2}$

- *A point (x_0, λ_0) on the equilibrium curve for the system $\dot{x} = F(x, \lambda)$ is called a* <u>pitchfork bifurcation</u> *if $\partial F/\partial\lambda|_{(x_0,\lambda_0)} = 0$, $\partial F/\partial x|_{(x_0,\lambda_0)} = 0$ and $d\lambda/dx$ changes sign on one branch of the equilibrium curve with distinct tangents, where $\lambda(x)$ is the solution of the equation $F(x, \lambda) = 0$.* $\boxed{2}$
  $\partial F/\partial\lambda|_{(x_0,\lambda_0)} = 0$ gives $x_0 = 0$ and subsequently $\partial F/\partial x|_{(x_0,\lambda_0)} = 0$ gives $\lambda_0 = 0$. Since $d\lambda/dx = 2x$ changes sign at $x_0 = 0$ and this branch has a different tangent than $x_1 = 0$, the point $(x_0, \lambda_0) = (0, 0)$ constitutes a pitchfork bifurcation.
- *A point (x_0, λ_0) on the equilibrium curve for the system $\dot{x} = F(x, \lambda)$ is called a* <u>transcritical bifurcation</u> *if $\partial F/\partial\lambda|_{(x_0,\lambda_0)} = 0$, $\partial F/\partial x|_{(x_0,\lambda_0)} = 0$ and in addition two and only two branches of the equilibrium curve pass through this point which have both distinct tangents at (x_0, λ_0).* $\boxed{2}$
  For $\gamma \neq 0$ we have $d\lambda/dx = 2x + \gamma$, which no longer changes sign at $x_0 = 0$. However, only two branches pass through this point and their tangents are distinct, such that $(x_0, \lambda_0) = (0, 0)$ constitutes a transcritical bifurcation.
- *A point (x_0, λ_0) on the equilibrium curve for the system $\dot{x} = F(x, \lambda)$ is called a* <u>turning point</u> *if $\partial F/\partial\lambda|_{(x_0,\lambda_0)} \neq 0$ and $\partial\lambda/\partial x$ changes sign at this point.* $\boxed{2}$

(ii) Defining $x_1 = x$ and : $x_2 = \dot{x}$ the differential equation converts into $\boxed{6}$

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\lambda(x_1^2 - 1)x_2 - 1 \end{aligned}$$

The fixed point is at $(x_1, x_2) = (0, 0)$. The Jacobian is computed to

$$J = \begin{pmatrix} 0 & 1 \\ -1 & \lambda \end{pmatrix},$$

such that the eigenvalues are

$$e_\pm(\lambda) = \frac{\lambda}{2} \pm \frac{\lambda}{2}\sqrt{\lambda^2 - 4}.$$

Therefore

$$\left.\begin{array}{ll} 0 < \lambda < 2 & \Rightarrow (0,0) \text{ is an unstable focus} \\ \lambda = 0 & \Rightarrow (0,0) \text{ is a centre} \\ -2 < \lambda < 0 & \Rightarrow (0,0) \text{ is a stable focus} \end{array}\right\} \equiv \text{Hopf bifurcation}$$

(iii) **Hopf bifurcation theorem:** *Let the point (0,0,λ), with $\lambda \in \mathbb{R}$, be a fixed point for the system* $\boxed{6}$

$$\dot{x}_1 = F_1(x_1, x_2, \lambda), \tag{15}$$
$$\dot{x}_2 = F_2(x_1, x_2, \lambda), \tag{16}$$

for all values of λ. If for a particular value of λ, say $\lambda = \tilde{\lambda}$,

i) *the eigenvalues $e_1(\lambda)$ and $e_2(\lambda)$ of the linearized system are purely imaginary, i.e. $e_1(\tilde{\lambda}) \in iR$ and $e_2(\tilde{\lambda}) \in iR$,*

ii) *the real part of the eigenvalues $\mathrm{Re}(e_1(\lambda)) = \mathrm{Re}(e_2(\lambda))$ satisfies*

$$\left.\frac{d}{d\lambda}\mathrm{Re}(e_{1/2}(\lambda))\right|_{\lambda=\tilde{\lambda}} > 0, \tag{17}$$

iii) *the origin is asymptotically stable for $\lambda = \tilde{\lambda}$,*

then the following statements hold:

a) *The point with $\lambda = \tilde{\lambda}$ is a bifurcation point of the system.*

b) *For $\lambda \in (\lambda_1, \tilde{\lambda})$ with some $\lambda_1 < \tilde{\lambda}$ the origin is a stable focus.*

c) *For $\lambda \in (\tilde{\lambda}, \lambda_2)$ with some $\lambda_2 > \tilde{\lambda}$ the origin is an unstable focus surrounded by a stable limit cycle whose size increases with λ.*

The eigenvalues for the Jacobian matrix with $\lambda \neq 0$ were computed to

$$e_\pm(\lambda) = \frac{\lambda}{2} \pm \frac{\lambda}{2}\sqrt{\lambda^2 - 4}.$$

i) for $\lambda = 0$ the eigenvalues are purely imaginary: $e_\pm = \pm i2$.

ii) we compute

$$\left.\frac{d}{d\lambda}\mathrm{Re}(e_{1/2}(\lambda))\right|_{\lambda=\tilde{\lambda}=0} = \frac{1}{2} > 0.$$

iii) We find $I = 0$ and therefore we can not use the stability index to establish point *iii*) of the theorem. $\boxed{\sum = 20}$

4. We have

$$H(x_1, x_2, p_1, p_2) = \frac{1}{2m}(p_1^2 + p_2^2) + \frac{k}{2}(x_1^2 + x_2^2).$$

(*i*) The equations of motion result to $\boxed{2}$

$$\begin{array}{ll} \frac{\partial H}{\partial x_1} = kx_1 = -\dot{p}_1 & \frac{\partial H}{\partial x_2} = kx_2 = -\dot{p}_2 \\ \frac{\partial H}{\partial p_1} = \frac{1}{m}p_1 = \dot{x}_1 & \frac{\partial H}{\partial p_2} = \frac{1}{m}p_2 = \dot{x}_2 \end{array}$$

(*ii*) The Poisson bracket is defined as $\boxed{6}$

$$\{f, g\} := \sum_{k=1}^{N/2} \frac{\partial f}{\partial q_k}\frac{\partial g}{\partial p_k} - \frac{\partial f}{\partial p_k}\frac{\partial g}{\partial q_k}.$$

For $L(x_1, x_2, p_1, p_2) = x_1 p_2 - x_2 p_1$ we compute

$$\begin{aligned} \dot{L} = \{L, H\} &= \frac{\partial L}{\partial x_1}\frac{\partial H}{\partial p_1} - \frac{\partial L}{\partial p_1}\frac{\partial H}{\partial x_1} + \frac{\partial L}{\partial x_2}\frac{\partial H}{\partial p_2} - \frac{\partial L}{\partial p_2}\frac{\partial H}{\partial x_2} \\ &= p_2\frac{1}{m}p_1 + x_2 k x_1 - p_1\frac{1}{m}p_2 - x_1 k x_2 = 0. \end{aligned}$$

Therfore L is conserved in time.
For $K(x_1, x_2, p_1, p_2) = \frac{1}{2m}(p_1^2 - p_2^2) + \frac{k}{2}(x_1^2 - x_2^2)$ we compute

$$\begin{aligned} \dot{K} = \{K, H\} &= \frac{\partial K}{\partial x_1}\frac{\partial H}{\partial p_1} - \frac{\partial K}{\partial p_1}\frac{\partial H}{\partial x_1} + \frac{\partial K}{\partial x_2}\frac{\partial H}{\partial p_2} - \frac{\partial K}{\partial p_2}\frac{\partial H}{\partial x_2} \\ &= kx_1\frac{1}{m}p_1 - \frac{1}{m}p_1 k x_1 - kx_2\frac{1}{m}p_2 + \frac{1}{m}p_2 k x_2 = 0 \end{aligned}$$

Therfore K is conserved in time.

(*iii*) **Jacobi-Poisson theorem:** *The Poisson bracket of two constants of motion $I_1(x_1, x_2, t)$ and $I_2(x_1, x_2, t)$ is also a constant of motion.* $\boxed{6}$
We compute

$$\begin{aligned} \{L, K\} &= \frac{\partial L}{\partial x_1}\frac{\partial K}{\partial p_1} - \frac{\partial L}{\partial p_1}\frac{\partial K}{\partial x_1} + \frac{\partial L}{\partial x_2}\frac{\partial K}{\partial p_2} - \frac{\partial L}{\partial p_2}\frac{\partial K}{\partial x_2} \\ &= p_2\frac{1}{m}p_1 + x_2 k x_1 + p_1\frac{1}{m}p_2 + x_1 k x_2 \\ &= \frac{2}{m}p_1 p_2 + 2kx_2 x_1 =: M \end{aligned}$$

Verify that M is conserved

$$\begin{aligned} \dot{M} = \{M, H\} &= \frac{\partial M}{\partial x_1}\frac{\partial H}{\partial p_1} - \frac{\partial M}{\partial p_1}\frac{\partial H}{\partial x_1} + \frac{\partial M}{\partial x_2}\frac{\partial H}{\partial p_2} - \frac{\partial M}{\partial p_2}\frac{\partial H}{\partial x_2} \\ &= 2kx_2\frac{1}{m}p_1 - \frac{2}{m}p_2 k x_1 + 2kx_1\frac{1}{m}p_2 - \frac{2}{m}p_1 k x_2 = 0. \end{aligned}$$

Therefore $\{L, K\} = M$ is conserved in time

(iv) The new quantity is not independent as $\boxed{6}$

$$H^2 = K^2 + \frac{1}{4}M^2 + \frac{k}{m}L^2$$

$\boxed{\sum = 20}$

5. (i) We can write

$$x_{n+1} = F(x_n) = (x_n + 2\lambda)(x_n - 6\lambda)$$

This means we have fixed points at $\boxed{1}$

$$x_f^{(1)} = -2\lambda \qquad \text{and} \qquad x_f^{(2)} = 6\lambda.$$

A fixed point x_f is stable iff $|F'(x_f)| < 1$. With $F'(x) = 2x - 4\lambda$ follows that
$x_f^{(1)}$ is stable for $|4\lambda + 4\lambda| < 1$, that is $\lambda < 1/8$.
$x_f^{(2)}$ is stable for $|6\lambda - 4\lambda| < 1$, that is $\lambda < 1/2$. $\boxed{3}$

(ii) A 2-cycle exists if $F(F(x)) = x$. Compute $\boxed{1}$

$$\begin{aligned} x &= (F(x) + 2\lambda)(F(x) - 6\lambda) \\ &= F^2(x) - 4\lambda F(x) - 12\lambda \\ &= (x + 2\lambda)^2(x - 6\lambda)^2 - 4\lambda(x + 2\lambda)(x - 6\lambda) - 12\lambda \\ &= x^4 - 8\lambda x^3 - 8\lambda^2 x^2 - 4\lambda x^2 + 96\lambda^3 x + 16\lambda^2 x + 144\lambda^4 + 48\lambda^3 - 12\lambda^2 \end{aligned}$$

Since the fixed point is a solution of this equation, we can factor out the term $F(x) - x$, i.e. $\boxed{5}$

$$F(F(x)) - x = (F(x) - x)p(x)$$

By polynomial devision we find $\boxed{4}$

$$p(x) = x^2 - 4\lambda x + x - 12\lambda^2 - 4\lambda + 1$$

This means for a two cycle to exist we require

$$x^2 - 4\lambda x + x - 12\lambda^2 - 4\lambda + 1 = 0.$$

Solving this quadratic equation for x gives

$$x_\pm = \frac{1}{2}\left(4\lambda - 1 \pm \sqrt{64\lambda^2 + 8\lambda - 3}\right)$$

For this to be real we require

$$64\lambda^2 + 8\lambda - 3 \geq 0.$$

Therefore the existence of a two cycle is ensured iff

$$\left[\lambda + \frac{1}{16}(1 + \sqrt{13})\right]\left[\lambda + \frac{1}{16}(1 - \sqrt{13})\right] \geq 0,$$

which means $\lambda \geq \frac{1}{16}(\sqrt{13} - 1)$. $\boxed{3}$

(iii) The 2 cycle is stable if for $G(x) = F(F(x))$

$$\left|G'(x)\right| < 1 \qquad \Leftrightarrow \qquad \left|F'(x_+)F'(x_-)\right| < 1$$

Compute

$$|(2x_+ - 4\lambda)(2x_- - 4\lambda)| = \left|-64\lambda^2 - 8\lambda + 4\right| < 1$$

This means

$$\left[\lambda + \frac{1}{16}(1+\sqrt{13})\right]\left[\lambda + \frac{1}{16}(1-\sqrt{13})\right] < 0 \quad \Rightarrow \lambda < \frac{1}{16}(\sqrt{13}-1)$$
$$\left[\lambda + \frac{1}{16}(1+\sqrt{21})\right]\left[\lambda + \frac{1}{16}(1-\sqrt{21})\right] > 0 \quad \Rightarrow \lambda > \frac{1}{16}(\sqrt{21}-1)$$

The domain of stability for the 2 cycle is empty. The two cycle is always unstable. $\boxed{3}$

$\boxed{\sum = 20}$