MA3608 Dynamical Systems II

Solutions and marking scheme for exam January 2013

INSTRUCTIONS: Full marks are obtained for correct answers to three of the five questions. Each question carries 25 marks.
(All questions are unseen, apart from definitions and theorems.)

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1. Consider the dynamical system of the form

$$\dot{x}_1 = x_2 - x_2^3 =: X_1,$$
$$\dot{x}_2 = -x_1 - x_2^2 =: X_2.$$

(i) In order to find the fixed points we need to solve $\boxed{2}$

$$0 = x_2 - x_2^3, \qquad \text{and} \qquad 0 = -x_1 - x_2^2.$$

We find three solutions

$$x_F^{(1)} = (0,0), \quad x_F^{(2)} = (-1,-1) \quad \text{and} \quad x_F^{(3)} = (-1,1).$$

(ii) **Linearization theorem:** *Consider a nonlinear system which possesses a simple linearization at some fixed point. Then in a neighbourhood of the fixed point the phase portraits of the linear system and its linearization are qualitatively equivalent, if the eigenvalues of the Jacobian matrix have a nonzero real part, i.e. the linearized system is not a centre.* $\boxed{1}$

We need to compute the Jacobian:

$$A = \left.\begin{pmatrix} \frac{\partial X_1}{\partial x_1} & \frac{\partial X_1}{\partial x_2} \\ \frac{\partial X_2}{\partial x_1} & \frac{\partial X_2}{\partial x_2} \end{pmatrix}\right|_{x_F}$$

We find

$$A(x_1,x_2) = \begin{pmatrix} 0 & 1-3x_2^2 \\ -1 & -2x_2 \end{pmatrix},$$

and therefore $\boxed{2}$

$$A(x_F^{(1)}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, A(x_F^{(2)}) = \begin{pmatrix} 0 & -2 \\ -1 & 2 \end{pmatrix}, A(x_F^{(3)}) = \begin{pmatrix} 0 & -2 \\ -1 & -2 \end{pmatrix}.$$

Next we compute the eigenvalues of $A(x_F)$ $\boxed{2}$

$$\det\left[A(x_F^{(1)}) - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] = 1 + \lambda^2 = 0 \Rightarrow \lambda_\pm^{(1)} = \pm i,$$

$$\det\left[A(x_F^{(2)}) - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] = -2 - 2\lambda + \lambda^2 = 0 \Rightarrow \lambda_\pm^{(2)} = 1 \pm \sqrt{3},$$

$$\det\left[A(x_F^{(3)}) - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] = -2 + 2\lambda + \lambda^2 = 0 \Rightarrow \lambda_\pm^{(3)} = -1 \pm \sqrt{3}.$$

This means the linearization theorem can not be applied for $x_F^{(1)}$, but for $x_F^{(2)}$ and $x_F^{(3)}$. We have a centre at $x_F^{(1)}$ and saddle points at $x_F^{(2)}$ and $x_F^{(3)}$. $\boxed{2}$

(iii) We compute the eigenvectors for $A(x_F^{(2)})$ and $A(x_F^{(3)})$: $\boxed{2}$

$$x_F^{(2)} : \begin{pmatrix} 0 & -2 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 1 \pm \sqrt{3} \begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow v_\pm^{(2)} = \begin{pmatrix} 1 \mp \sqrt{3} \\ 1 \end{pmatrix}$$

$$x_F^{(3)} : \begin{pmatrix} 0 & -2 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = -1 \pm \sqrt{3} \begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow v_\pm^{(3)} = \begin{pmatrix} -1 \mp \sqrt{3} \\ 1 \end{pmatrix}$$

This means the matrix $U = \{v_+, v_-\}$ can be used to transform A into the Jordan normal form. Therefore

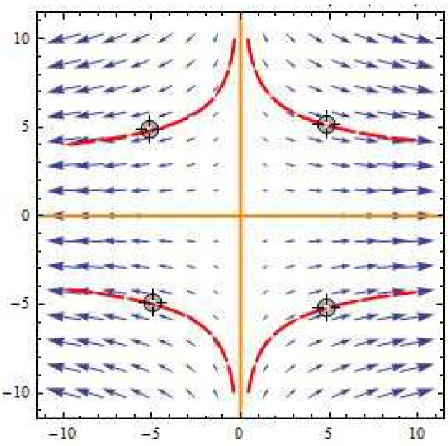
$$\left(U^{(2)}\right)^{-1} A(x_F^{(2)}) U^{(2)} = \begin{pmatrix} \lambda_+^{(2)} & 0 \\ 0 & \lambda_-^{(2)} \end{pmatrix} \quad \text{with } U^{(2)} = \left\{v_+^{(2)}, v_-^{(2)}\right\},$$

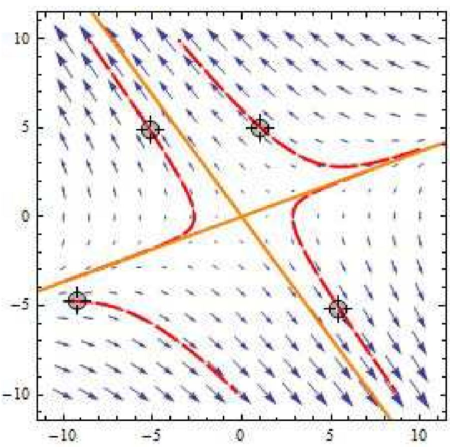
$$\left(U^{(3)}\right)^{-1} A(x_F^{(3)}) U^{(3)} = \begin{pmatrix} \lambda_+^{(3)} & 0 \\ 0 & \lambda_-^{(3)} \end{pmatrix} \quad \text{with } U^{(3)} = \left\{v_+^{(3)}, v_-^{(3)}\right\}.$$

(This does not have to be computed as it is known to be correct, but can be used as consistency check.) We use U to transform the separatrices of the phase portrait for a saddle point belonging to a linearized system in Jordan normal. For $x_F^{(2)}$ we compute $\boxed{3}$

$$U^{(2)} \begin{pmatrix} a \\ 0 \end{pmatrix} = \begin{pmatrix} (1-\sqrt{3})a \\ a \end{pmatrix}, \quad U^{(2)} \begin{pmatrix} 0 \\ a \end{pmatrix} = \begin{pmatrix} (1+\sqrt{3})a \\ a \end{pmatrix}, \; a \in \mathbb{R}.$$

The local phase portraits for the linearized system related to $\{\{\lambda_+^{(2)}, 0\}, \{0, \lambda_-^{(2)}\}\}$ and $A(x_F^{(2)})$, respectively, results to:



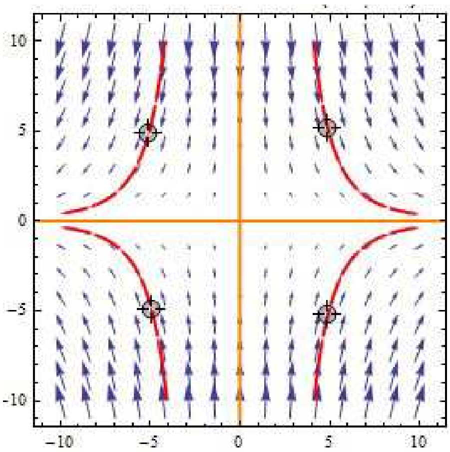


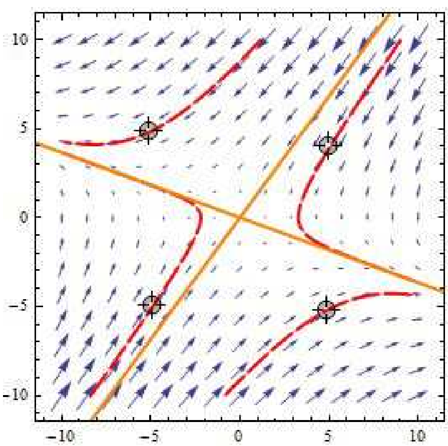
(Full marks are given for a qualitatively correct transformation, i.e. for correct transformed directrices together with a few lines in between with the correct direction of time indicated.)

For $x_F^{(3)}$ we compute 3

$$U^{(3)}\begin{pmatrix} a \\ 0 \end{pmatrix} = \begin{pmatrix} (-1-\sqrt{3})a \\ a \end{pmatrix}, \quad U^{(3)}\begin{pmatrix} 0 \\ a \end{pmatrix} = \begin{pmatrix} (-1+\sqrt{3})a \\ a \end{pmatrix}, \; a \in \mathbb{R}.$$

The local phase portraits for the linearized system related to $\{\{\lambda_+^{(3)}, 0\}, \{0, \lambda_-^{(3)}\}\}$ and $A(x_F^{(3)})$, respectively, results to:





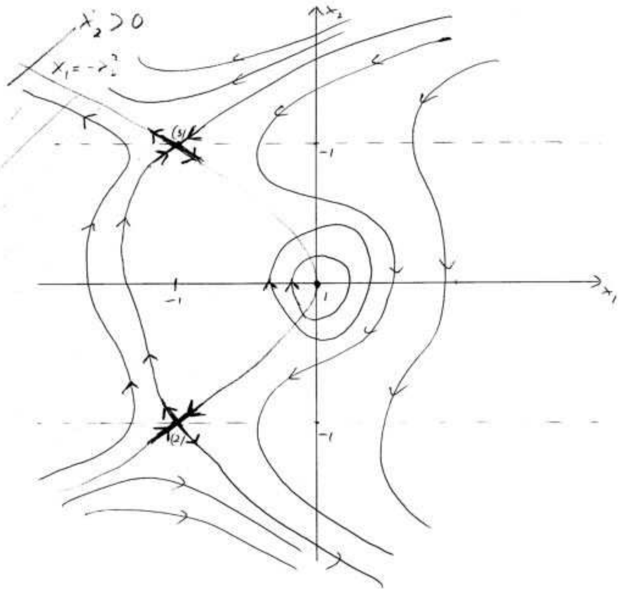
(*iv*) We have $\dot{x}_1 > 0$ for $x_2(1-x_2^2) > 0$, i.e. $x_2 > 0 \wedge x_2^2 < 1 \vee x_2 < 0 \wedge x_2^2 > 1$. 2

We have $\dot{x}_2 > 0$ for $-x_1 - x_2^2 > 0$, i.e. $-x_1 > x_2^2$.

We compute the isoclines to

$$\frac{dx_2}{dx_1} = \frac{\dot{x}_2}{\dot{x}_1} = \frac{-x_1 - x_2^2}{x_2 - x_2^3} = \begin{cases} 0 & \text{for } \; x_1 = -x_2^2 \\ \infty & \text{for } \; x_2 = 0, x_2 = \pm 1 \end{cases}$$

(*v*) Assembling the information from (*i*)−(*iv*) we obtain the following phase portrait:



$\boxed{6}$

$\boxed{\sum = 25}$

2. (i) The Jacobian is computed to $\boxed{3}$

$$A((0,0)) = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

The eigenvalues are $\lambda_\pm = 1 \pm i$ and therefore the origin is an unstable focus.

(ii) With $x_1 = r\cos\vartheta$ and $x_2 = r\sin\vartheta$ we obtain $\boxed{5}$

$$\dot{x}_1 = \dot{r}\cos\vartheta - r\sin\vartheta\dot{\vartheta} = r\cos\vartheta - r\sin\vartheta - r\cos\vartheta(r^2 + 4r^2\sin^2\vartheta) \qquad (1)$$

$$\dot{x}_2 = \dot{r}\sin\vartheta + r\cos\vartheta\dot{\vartheta} = r\cos\vartheta + r\sin\vartheta - r\sin\vartheta(r^2) \qquad (2)$$

where $x_1^2 + x_2^2 = r^2$. Computing (1) $\times\cos\vartheta$ + (2) $\times$ $\sin\vartheta$ gives

$$\begin{aligned} \dot{r} &= r - r^3 - 4r^3\sin^2\vartheta\cos^2\vartheta \\ &= r[1 - r^2 - r^2\sin^2(2\vartheta)]. \end{aligned}$$

Next we compute (2) $\times$ $\cos\vartheta-$ (1) $\times\sin\vartheta$:

$$r\dot{\vartheta} = r + 4r^3\cos\vartheta\sin^3\vartheta$$

Dividing by r gives

$$\dot{\vartheta} = 1 + 2r^2\sin^2\vartheta\sin 2\vartheta.$$

For the fixed point we have $\dot{\vartheta} = 0$ and $\dot{r} = 0$. Therefore $\boxed{1}$

$$1 + 2r^2\sin^2\vartheta\sin(2\vartheta) = 0 \quad \text{and} \quad 1 - r^2 - r^2\sin^2(2\vartheta) = 0,$$

which means we have to have

$$1 + \sin^2(2\vartheta) + 2\sin^2\vartheta\sin(2\vartheta) = 0.$$

As there is no real solution to this, by hint, the origin is the only fixed point.

(*iii*) **Poincaré-Bendixson theorem:** *Let φ_t be a flow for the system $\dot{\vec{x}} = \vec{F}(\vec{x})$ and let $\mathcal{D}$ be a closed, bounded and connected set $\mathcal{D} \in \mathbb{R}^2$, such that $\varphi_t(\mathcal{D}) \subset \mathcal{D}$ for all time. Furthermore $\mathcal{D}$ does not contain any fixed point. Then there exists at least one limit cycle in $\mathcal{D}$.* $\boxed{2}$

For $\mathcal{D} = \left\{(r, \vartheta) : \frac{1}{2} \leq r \leq 2\right\}$ we compute $\boxed{4}$

$$\begin{aligned} \dot{r}(2) &= 2[1 - 4 - 4\sin^2(2\vartheta)] < 0, \\ \dot{r}\left(\frac{1}{2}\right) &= 2[1 - \frac{1}{4} - \frac{1}{4}\sin^2(2\vartheta)] > 0. \end{aligned}$$

Therefore any trajectory which enters the region $\mathcal{D}$ can never leave it. Since there is no fixed point in $\mathcal{D}$, by (*ii*), we can employ the Poincaré-Bendixson theorem to deduce that there is at least one limit cycle in $\mathcal{D}$.

(*iv*) In order to find the limit cycle domain we require $\dot{r} > 0 \; \forall \vartheta$ on the inner boundary, which is equivalent to $\boxed{6}$

$$1 - r^2 - r^2 \sin^2(2\vartheta) > 0 \qquad \forall \vartheta.$$

This means that for $r \neq 0$ we demand

$$r^2 < \frac{1}{1 + \sin^2(2\vartheta)} \qquad \forall \vartheta.$$

If we replace the right hand side by its minimum the inequality will hold still hold for all values of ϑ.

$$r^2 < \min\left[\frac{1}{1 + \sin^2(2\vartheta)}\right] = \frac{1}{2} \qquad \forall \vartheta.$$

On the outer boundary we require $\dot{r} < 0 \; \forall \vartheta$ on the inner boundary, which is equivalent to

$$r^2 > \frac{1}{1 + \sin^2(2\vartheta)} \qquad \forall \vartheta.$$

We replace the right hand side by its maximum.

$$r^2 > \max\left[\frac{1}{1 + \sin^2(2\vartheta)}\right] = 1 \qquad \forall \vartheta.$$

This means we can now define a new closed, bounded and connected set

$$\mathcal{D}^{\varepsilon} = \left\{(r, \vartheta) : \frac{1}{\sqrt{2}} - \varepsilon \leq r \leq 1 + \varepsilon\right\},$$

where $0 < \varepsilon \ll 1$. Since $\dot{r} > 0$ on the inner boundary and $\dot{r} < 0$ on the outer boundary, this means that trajectories which enter the domain $\mathcal{D}^{\varepsilon}$ do not leave it anymore. This implies by the Poincaré-Bendixson theorem that there is at least one periodic orbit in $\mathcal{D}^{\varepsilon}$. Since $r = 1/\sqrt{2}$ and $r = 1$ are no trajectories of the system, the above statements are also true for $\varepsilon = 0$, that is we may consider the new optimized domain

$$\mathcal{D} = \left\{(r, \vartheta) : r_{\min} = \frac{1}{\sqrt{2}} \leq r \leq 1 = r_{\max}\right\}.$$

(*v*) **Def.:** *The* <u>*ω-limit set*</u> *(or positive limit set)* $L_\omega(\vec{x})$ *of a point* $\vec{x}$ *contains those points which are approached by the trajectory through* $\vec{x}$ *as* $t \to \infty$*, that is* 3

$$L_\omega(\vec{x}) = \{\vec{y} \in \mathbb{R}^n : \exists \text{ a sequence of times } t_n \text{ with } t_n \to \infty, \\ \text{such that } \lim_{n\to\infty} \varphi_{t_n}(\vec{x}) = \vec{y}\Big\}$$

Def.: *The* <u>*α-limit set*</u> *(or negative limit set)* $L_\alpha(\vec{x})$ *of a point* $\vec{x}$ *contains those points which are approached by the trajectory through* $\vec{x}$ *as* $t \to -\infty$*, that is*

$$L_\alpha(\vec{x}) = \{\vec{y} \in \mathbb{R}^n : \exists \text{ a sequence of times } t_n \text{ with } t_n \to -\infty, \\ \text{such that } \lim_{n\to\infty} \varphi_{t_n}(\vec{x}) = \vec{y}\Big\}$$

Def.: *A closed orbit* ϕ *is a* <u>*limit cycle*</u> *if* ϕ *is a subset of an* α *or* ω*-limit set for some point* $\vec{x} \notin \phi$.
The ω limit set for $\mathcal{D}_s$ is one of the limit cycles in $\mathcal{D}_s$. 1

$\sum = 25$

3. (*i*) *Bifurcation theory* investigates how the number of steady solutions of systems of the type $\dot{x} = F(x, \lambda)$ depend on the parameter λ. A *bifurcation* occurs if the solution of $\dot{x} = F(x, \lambda)$ changes its qualitative behaviour as the parameter λ varies. 1

- *A point* (x_0, λ_0) *on the equilibrium curve for the system* $\dot{x} = F(x, \lambda)$ *is called a* <u>*pitchfork bifurcation*</u> *if* $\partial F/\partial\lambda|_{(x_0,\lambda_0)} = 0$, $\partial F/\partial x|_{(x_0,\lambda_0)} = 0$ *and* $d\lambda/dx$ *changes sign on one branch of the equilibrium curve with distinct tangents, where* $\lambda(x)$ *is the solution of the equation* $F(x, \lambda) = 0$. 1
  $\partial F/\partial\lambda|_{(x_0,\lambda_0)} = 0$ gives $x_0 = 0$ and subsequently $\partial F/\partial x|_{(x_0,\lambda_0)} = 0$ gives $\lambda_0 = 0$. Since $d\lambda/dx = 2x$ changes sign at $x_0 = 0$ and this branch has a different tangent than $x_1 = 0$, the point $(x_0, \lambda_0) = (0, 0)$ constitutes a pitchfork bifurcation.
- *A point* (x_0, λ_0) *on the equilibrium curve for the system* $\dot{x} = F(x, \lambda)$ *is called a* <u>*transcritical bifurcation*</u> *if* $\partial F/\partial\lambda|_{(x_0,\lambda_0)} = 0$, $\partial F/\partial x|_{(x_0,\lambda_0)} = 0$ *and in addition two and only two branches of the equilibrium curve pass through this point which have both distinct tangents at* (x_0, λ_0). 1
  For $\gamma \neq 0$ we have $d\lambda/dx = 2x + \gamma$, which no longer changes sign at $x_0 = 0$. However, only two branches pass through this point and their tangents are distinct, such that $(x_0, \lambda_0) = (0, 0)$ constitutes a transcritical bifurcation.
- *A point* (x_0, λ_0) *on the equilibrium curve for the system* $\dot{x} = F(x, \lambda)$ *is called a* <u>*turning point*</u> *if* $\partial F/\partial\lambda|_{(x_0,\lambda_0)} \neq 0$ *and* $\partial\lambda/\partial x$ *changes sign at this point.* 1

(*ii*) To find the fixed points we need to solve 2

$$0 = 1 - (1 + \lambda)x_1 + \gamma x_1^2 x_2 \quad \text{and} \quad 0 = \lambda x_1 - \gamma x_1^2 x_2,$$

which gives as the only solution

$$x_1 = 1 \quad \text{and} \quad x_2 = \lambda/\gamma \Rightarrow x_F = (1, \lambda/\gamma)$$

The Jacobian is computed to

$$A(x_1, x_2) = \begin{pmatrix} -1 - \lambda + 2\gamma x_1 x_2 & \gamma x_1^2 \\ \lambda - 2\gamma x_1 x_2 & -\gamma x_1^2 \end{pmatrix}.$$

therefore $\boxed{4}$

$$A(x_F) = \begin{pmatrix} \lambda - 1 & \gamma \\ -\lambda & -\gamma \end{pmatrix}.$$

The eigenvalues are computed to

$$e_{\pm} = \frac{\lambda - \gamma - 1}{2} \pm \frac{1}{2}\sqrt{(1 - \lambda + \gamma)^2 - 4\gamma}$$

For instance: When $\lambda = \gamma + 1$ we have a centre since $e_{\pm} = \pm i\sqrt{\gamma} \in i\mathbb{R}$. When $\lambda = \gamma + 1 + 2\sqrt{\gamma}$ we have $e_- = e_+ = \sqrt{\gamma} > 0$ which is an unstable improper node as the Jordan normal form is not diagonal. When $\lambda = \gamma + 1 - 2\sqrt{\gamma}$ we have $e_- = e_+ = -\sqrt{\gamma} < 0$ which is a stable improper node as the Jordan normal form is not diagonal. (Any other three of the ten possibilities are also fine.) $\boxed{3}$

(*iii*) **Hopf bifurcation theorem:** *Let the point (0,0,λ), with $\lambda \in \mathbb{R}$, be a fixed point for the system* $\boxed{4}$

$$\dot{x}_1 = F_1(x_1, x_2, \lambda),$$
$$\dot{x}_2 = F_2(x_1, x_2, \lambda),$$

for all values of λ. If for a particular value of λ, say $\lambda = \tilde{\lambda}$,

i) *the eigenvalues $e_1(\lambda)$ and $e_2(\lambda)$ of the linearized system are purely imaginary, i.e. $e_1(\tilde{\lambda}) \in iR$ and $e_2(\tilde{\lambda}) \in iR$,*

ii) *the real part of the eigenvalues $\mathrm{Re}(e_1(\lambda)) = \mathrm{Re}(e_2(\lambda))$ satisfies*

$$\frac{d}{d\lambda} \mathrm{Re}(e_{1/2}(\lambda)) \bigg|_{\lambda = \tilde{\lambda}} > 0,$$

iii) *the origin is asymptotically stable for $\lambda = \tilde{\lambda}$,*

then the following statements hold:

a) *The point with $\lambda = \tilde{\lambda}$ is a bifurcation point of the system.*

b) *For $\lambda \in (\lambda_1, \tilde{\lambda})$ with some $\lambda_1 < \tilde{\lambda}$ the origin is a stable focus.*

c) *For $\lambda \in (\tilde{\lambda}, \lambda_2)$ with some $\lambda_2 > \tilde{\lambda}$ the origin is an unstable focus surrounded by a stable limit cycle whose size increases with λ.*

(*iv*) For $\tilde{\lambda} = \gamma + 1$ we have $e_{\pm} = \pm i\sqrt{\gamma} \in i\mathbb{R}$, which means a centre. For $\lambda < \tilde{\lambda}$ we have a stable fixed point and for $\lambda > \tilde{\lambda}$ we have an unstable fixed point. This means as λ varies from $\lambda < \tilde{\lambda}$ to $\lambda > \tilde{\lambda}$ the fixed point changes its characteristic behaviour from stable, to centre, to unstable, i.e. it undergoes a Hopf bifurcation. $\boxed{3}$

(*v*) The Jacobian for the linearized system in Jordan normal form is $\boxed{5}$

$$J = \begin{pmatrix} 0 & -\sqrt{\gamma} \\ \sqrt{\gamma} & 0 \end{pmatrix}.$$

Therefore the dynamical system takes on the form

$$\dot{x}_1 = -\sqrt{\gamma} x_2, \quad \text{and} \quad \dot{x}_2 = \sqrt{\gamma} x_1,$$

such that

$$\ddot{x}_1 = -\sqrt{\gamma}\dot{x}_2 = -\gamma x_1.$$

The solution of this equation is $x_1 = c_1 \sin(\sqrt{\gamma} x_1) + c_2 \cos(\sqrt{\gamma} x_1)$. Therefore the period is estimated to

$$T = \frac{2\pi}{\sqrt{\gamma}}.$$

$\boxed{\sum = 25}$

4. (*i*) **Def.:** *A system of differential equations on* $\mathbb{R}^2$ *is said to be a* <u>Hamiltonian system</u> *with one degree of freedom if there exists a twice continuously differentiable function* $H(x_1, x_2)$ *such that* $\boxed{3}$

$$\dot{x}_1 = \frac{\partial H}{\partial x_2} \qquad \text{and} \qquad \dot{x}_2 = -\frac{\partial H}{\partial x_1}. \tag{3}$$

The equations (3) are said to be the *equations of motions* corresponding to the Hamiltonian H. When H does not depend explicitly on the time t, i.e. it is of the form $H(x_1(t), x_2(t))$ and not $H(x_1(t), x_2(t), t)$, the system is called *autonomous.*

(*ii*) *Any nondegenerate fixed point of a Hamiltonian system is either a saddle point or a centre.* $\boxed{7}$

Proof: We compute the Jacobian matrix to

$$A = \begin{pmatrix} \frac{\partial^2 H}{\partial x_1 \partial x_2} & \frac{\partial^2 H}{\partial x_2^2} \\ -\frac{\partial^2 H}{\partial x_1^2} & -\frac{\partial^2 H}{\partial x_1 \partial x_2} \end{pmatrix}\Bigg|_{\vec{x}_f} =: \begin{pmatrix} H_{12} & H_{22} \\ -H_{11} & -H_{12} \end{pmatrix}\Bigg|_{\vec{x}_f}.$$

The eigenvalues are then obtained from

$$\det(A - \lambda\mathbb{I}) = (H_{12} - \lambda)(-H_{12} - \lambda) + H_{11}H_{22} = 0,$$

such that

$$\lambda^2 = -H_{11}H_{22} + H_{12}^2.$$

When the fixed point is nondegenerate we only have the two possibilities

$$H_{12}^2 - H_{11}H_{22} \begin{cases} > 0 & \equiv \text{ real eigenvalues of opposite sign } \equiv \text{ saddle point} \\ < 0 & \equiv \text{ purely imaginary eigenvalues } \equiv \text{ centre} \end{cases},$$

which is what we wanted to prove. $\square$

(*iii*) *A dynamical system* $\boxed{2}$

$$\dot{x}_1 = F_1(x_1, x_2) \qquad \text{and} \qquad \dot{x}_2 = F_2(x_1, x_2),$$

is a Hamiltonian system if and only if

$$\operatorname{div} \vec{F} = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} = 0.$$

We compute $\boxed{5}$

$$\operatorname{div} \vec{F} = x_2^2 x_1 (2\alpha + 3\beta) e^{(\alpha-1)x_1^2} + \alpha(5\alpha - 2) x_2^2 x_1^3 e^{(\alpha-1)x_1^2} + 2 + \gamma = 0.$$

Therefore the system is a Hamiltonian system when $\alpha = 2/5$, $\beta = -4/15$, $\gamma = -2$ or $\alpha = 0$, $\beta = 0$, $\gamma = -2$.

(*iv*) **Def.:** *A Hamiltonian system which is of the form* $\boxed{2}$

$$H(x_1, x_2) = \frac{1}{2} x_2^2 + V(x_1),$$

where $V(x_1)$ is a function which only depends on x_1 and not x_2 is called a *potential system* *with* *potential (function)* *$V(x_1)$.*

From the definition in (*i*) follows $\boxed{4}$

$$\begin{aligned} \dot{x}_1 &= \frac{\partial H}{\partial x_2} = x_2 \Rightarrow H(x_1, x_2) = \frac{1}{2} x_2^2 + f(x_1) \\ \dot{x}_2 &= -\frac{\partial H}{\partial x_1} = x_1^2 + \frac{5}{(3 - x_1)(2 + x_1)} \Rightarrow H(x_1, x_2) = \frac{x_1^3}{3} + \ln \frac{2 + x_1}{3 - x_1} + \tilde{f}(x_2). \end{aligned}$$

with $f(x_1)$ and $\tilde{f}(x_2)$ some arbitrary functions. Therefore

$$H(x_1, x_2) = \frac{1}{2} x_2^2 + \frac{x_1^3}{3} + \ln \frac{2 + x_1}{3 - x_1} + c,$$

such that the potential is

$$V(x_1) = \frac{x_1^3}{3} + \ln \frac{2 + x_1}{3 - x_1} + c.$$

From $V(0) = 0$ follows $c = -\ln(2/3)$.

(*v*) **Jacobi-Poisson theorem:** *The Poisson bracket of two constants of motion $I_1(x_1, x_2, t)$ and $I_2(x_1, x_2, t)$* $\boxed{2}$

$$\{I_1, I_2\} := \frac{\partial\ I_1}{\partial x_1} \frac{\partial I_2}{\partial x_2} - \frac{\partial\ I_1}{\partial x_2} \frac{\partial I_2}{\partial x_1}.$$

is also a constant of motion. $\boxed{\sum = 25}$

5. (i) The fixed points are found from $\boxed{1}$

$$F(x) = x \qquad \Leftrightarrow \qquad 3x - 6\lambda x + 2\lambda x^2 = x$$

This means we have fixed points at $\boxed{5}$

$$x_f^{(1)} = 0 \qquad \text{and} \qquad x_f^{(2)} = 3 - \frac{1}{\lambda}.$$

A fixed point x_f is stable iff $|F'(x_f)| < 1$.
With $F'(x) = 3 - 6\lambda + 4\lambda x$ follows that
$x_f^{(1)}$ is stable for $|3 - 6\lambda| < 1 \Leftrightarrow (\lambda - 1/3)(\lambda - 2/3) < 0$, that is $1/3 < \lambda < 2/3$.
$x_f^{(2)}$ is stable for $|3 - 6\lambda + 4(3\lambda - 1)| < 1, \Leftrightarrow 12\lambda(3\lambda - 1) < 0$ that is $0 < \lambda < 1/3$.

(ii) A k-cycle exists iff $F^k(x) = x$. For the 2-cycle we therefore have to evaluate $F(F(x)) = x$ $\boxed{6}$

$$\begin{aligned} x &= 3F(x) - 6\lambda F(x) + 2\lambda F^2(x), \\ &= 3(3x - 6\lambda x + 2\lambda x^2) - 6\lambda(3x - 6\lambda x + 2\lambda x^2) + 2\lambda(3x - 6\lambda x + 2\lambda x^2)^2, \\ &= 8\lambda^3 x^4 - 48\lambda^3 x^3 + 24\lambda^2 x^3 + 72\lambda^3 x^2 - 84\lambda^2 x^2 + 24\lambda x^2 + 36\lambda^2 x - 36\lambda x + 9x. \end{aligned}$$

Since the fixed point is a solution of this equation, we can factor out the term $F(x) - x$. Not knowing the answer this can be done by polynomial division, but in this case it is sufficient to verify that: $\boxed{4}$

$$\begin{aligned} &(F(x) - x)\left(4\lambda^2 x^2 + \left(8\lambda - 12\lambda^2\right) x + 4 - 6\lambda\right) \\ &= 8\lambda^3 x^4 - 48\lambda^3 x^3 + 24\lambda^2 x^3 + 72\lambda^3 x^2 - 84\lambda^2 x^2 + 24\lambda x^2 + 36\lambda^2 x - 36\lambda x + 9x = 0 \end{aligned}$$

This means we require $\boxed{4}$

$$4\lambda^2 x^2 + \left(8\lambda - 12\lambda^2\right) x + 4 - 6\lambda = 0$$

for a two cycle to exist. Solving this quadratic equation gives

$$x_\pm = \frac{3\lambda^2 - 2\lambda \pm \sqrt{3}\sqrt{3\lambda^4 - 2\lambda^3}}{2\lambda^2}$$

For this to be real we require

$$3\lambda - 2 \geq 0.$$

Therefore the existence of a two cycle is ensured iff

$$\lambda \geq \frac{2}{3}.$$

(iii) The 2 cycle is stable when $\boxed{3}$

$$\left|G'(x)\right| < 1 \qquad \Leftrightarrow \qquad \left|F'(x_+)F'(x_-)\right| < 1$$

where $G(x) = F(F(x))$. We compute therefore

$$\begin{aligned} |(3-6\lambda+4\lambda x_+)(3-6\lambda+4\lambda x_-)| &= \left|1+24\lambda-36\lambda^2\right| < 1 \\ &\Leftrightarrow 24\lambda(3\lambda-2)\left(18\lambda^2-12\lambda-1\right) < 0 \end{aligned}$$

This means the two cycle is stable in the regime $\boxed{2}$

$$\frac{2}{3} < \lambda < \frac{1}{6}(2+\sqrt{6})$$

and unstable for $\lambda > \frac{1}{6}(2+\sqrt{6})$ $\boxed{\sum = 25}$