Solutions I

1) i)

(1) $x_1(\alpha - \beta x_2) = 0 \Rightarrow x_1 = 0$ or $x_2 = \alpha/\beta$

(2) $x_2(-\gamma + \delta x_1) = 0 \Rightarrow x_2 = 0$ or $x_1 = \gamma/\delta$

$x_1 = 0$ in (2) $\Rightarrow x_2 = 0$

$x_2 = \alpha/\beta$ in (2) $\Rightarrow x_1 = \gamma/\delta$

$\Rightarrow$ 2 fixed points: $\vec{x}_f^{(1)} = (0,0)$; $\vec{x}_f^{(2)} = \left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right)$

ii) (1) $x_2 = -5$

(2) $\cos x_1 = 0 \Rightarrow x_1^{(n)} = (2n-1)\frac{\pi}{2}$ $n \in \mathbb{Z}$ $\Rightarrow$ infinitely many f.p. $\vec{x}_f^{(n)} = \left((2n-1)\frac{\pi}{2}, -5\right)$

iii) (1) $e^{x_1} + \alpha x_2 + 5x_1 - \beta = 0$

(2) $x_1^3 = 0 \Rightarrow x_1 = 0$ into (1) $\Rightarrow 1 - \beta + \alpha x_2 = 0$

a) $\alpha \neq 0 \Rightarrow x_2 = \frac{\beta - 1}{\alpha} \Rightarrow$ one fixed point: $\vec{x}_f = \left(0, \frac{\beta-1}{\alpha}\right)$

b) $\alpha = 0, \beta = 1 \Rightarrow$ infinitely many f.p.: $\vec{x}_f = (0, x_2)$

c) $\alpha = 0, \beta \neq 1 \Rightarrow$ no fixed points

iv)

(1) $x_2(2 - x_2 - 2x_1) = 0 \Rightarrow x_2 = 0$ into (2) $\Rightarrow x_1 = 0$ or $x_1 = 2$

(2) $x_1(2 - x_1 - 2x_2) = 0 \Rightarrow x_1 = 0$ into (2) $\Rightarrow x_2 = 0$ or $x_2 = 2$

or $2 - x_2 - 2x_1 = 0 \wedge 2 - x_1 - 2x_2 = 0 \Rightarrow -2 + 3x_2 = 0 \Rightarrow x_2 = \frac{2}{3}$

$\Rightarrow \frac{4}{3} - 2x_1 = 0 \Rightarrow x_1 = \frac{2}{3}$

$\Rightarrow$ four fixed points: $\vec{x}_f^{(1)} = (0,0)$; $\vec{x}_f^{(2)} = (0,2)$; $\vec{x}_f^{(3)} = (2,0)$; $\vec{x}_f^{(4)} = \left(\frac{2}{3}, \frac{2}{3}\right)$

2) i)

$A = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}$ $\det A_\lambda = 0 = \lambda^2 - 3\lambda + 2 \Rightarrow e_1 = 2$ with $\vec{V}_1 = (1,2)$

$e_2 = 1$ with $\vec{V}_2 = (1,1)$

$\Rightarrow U = (V_1, V_2) = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \Rightarrow U^{-1} = -\begin{pmatrix} 1 & -1 \\ -2 & 1 \end{pmatrix}$

$\Rightarrow J = U^{-1} A U = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \equiv$ unstable node

recall $U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\Rightarrow U^{-1} = \frac{1}{\det U}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

$A = \begin{pmatrix} -2 & 2 \\ -2 & 3 \end{pmatrix}$ $\quad e_1 = 2$ with $\vec{V}_1 = (1, 2)$

$\quad e_2 = -1$ with $\vec{V}_2 = (2, 1)$

$\Rightarrow U = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ $\quad U^{-1} = -\frac{1}{3}\begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}$ $\quad \Rightarrow J = U^{-1} A U = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$ $\equiv$ saddlepoint

$A = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}$ $\quad e_1 = e_2 = 2$ $\quad \vec{V}_1 = \vec{V}_2 = (-1, 1)$

$\Rightarrow U = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad U^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ $\quad \Rightarrow J = U^{-1} A U = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ $\equiv$ unstable improper node

fixed points:

(1) $x_1 (12 - 3x_1 - 4x_2) = 0$ $\quad \Rightarrow x_1 = 0$ into (2) $\Rightarrow$ $x_2 = 0$ or $x_2 = 4$

(2) $x_2 (4 - 2x_1 - x_2) = 0$ $\quad \Rightarrow x_2 = 0$ into (1) $\Rightarrow$ $x_1 = 0$ or $x_1 = 4$

or $12 - 3x_1 - 4x_2 = 0 \ \wedge \ 4 - 2x_1 - x_2 = 0$ $\quad \Rightarrow -4 + 5x_1 = 0$ $\quad \Rightarrow x_1 = \frac{4}{5}$

$\Rightarrow x_2 = \frac{12}{5}$

$\Rightarrow$ four fixed points:

$\vec{X}_f^{(1)} = (0, 0)$ $\quad \vec{X}_f^{(2)} = (0, 4)$ $\quad \vec{X}_f^{(3)} = (4, 0)$ $\quad \vec{X}_f^{4} = \left(\frac{4}{5}, \frac{12}{5}\right)$

linearization:

$$A(x_1, x_2) = \begin{pmatrix} (12 - 6x_1 - 4x_2) & -4x_1 \\ -2x_2 & (4 - 2x_1 - 2x_2) \end{pmatrix}$$

$A(\vec{X}_f^{(1)}) = \begin{pmatrix} 12 & 0 \\ 0 & 4 \end{pmatrix}$ $\equiv$ unstable node $\because$ $e_1 = 12$ $\quad e_2 = 4$

$\vec{V}_1 = (1, 0)$ $\quad \vec{V}_2 = (0, 1)$

$A(\vec{X}_f^{(2)}) = \begin{pmatrix} -4 & 0 \\ -8 & -4 \end{pmatrix}$ $\equiv$ stable improper node $\because$ $e_1 = e_2 = -4$

$\vec{V}_1 = \vec{V}_2 = (0, 1)$

$J = U^{-1} A U = \begin{pmatrix} -4 & 1 \\ 0 & -4 \end{pmatrix}$ with $U = \begin{pmatrix} 0 & -1/8 \\ 1 & 0 \end{pmatrix}$

$$A\left(x_f^{(3)}\right) = \begin{pmatrix} -12 & -16 \\ 0 & -4 \end{pmatrix} \equiv \text{stable node} \quad \because \quad e_1 = -12 \quad e_2 = -4$$

$$\vec{V}_1 = (1,0) \quad \vec{V}_2 = (-2,1)$$

$$J = U^{-1} A U = \begin{pmatrix} -12 & 0 \\ 0 & -4 \end{pmatrix} \quad \text{with } U = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$$

$$A\left(x_f^{(4)}\right) = \begin{pmatrix} -\frac{12}{5} & -\frac{16}{5} \\ -\frac{24}{5} & -\frac{12}{5} \end{pmatrix} \equiv \text{saddle point} \quad \because \quad e_\pm = -\frac{4}{5}\left(3 \pm 2\sqrt{6}\right)$$

$$V_\pm = \left(\pm\sqrt{\tfrac{2}{3}}, 1\right)$$

$$J = U^{-1} A U = \begin{pmatrix} -\frac{4}{5}\left(3-2\sqrt{6}\right) & 0 \\ 0 & -\frac{4}{5}\left(3+2\sqrt{6}\right) \end{pmatrix} \quad \text{with } U = \left(V_-, V_+\right)$$

$$= \begin{pmatrix} 1.519\ldots & 0 \\ 0 & -6.319\ldots \end{pmatrix}$$

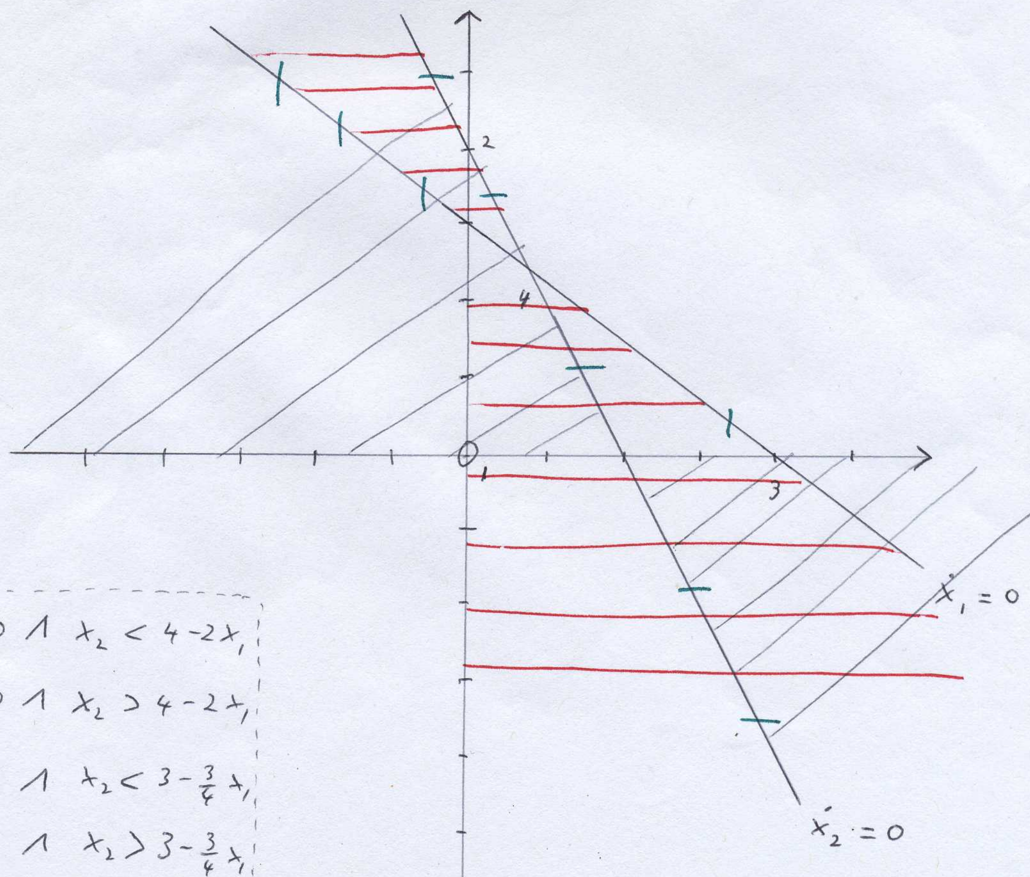
The linearisation theorem applies in all fixed poin

<u>isoclines:</u>

$$\frac{x_2\left(4 - 2x_1 - x_2\right)}{x_1\left(12 - 3x_1 - 4x_2\right)} = k$$

$k \to 0 \qquad x_2 = 0 \quad \text{or} \quad 4 - 2x_1 - x_2 = 0 \quad \Rightarrow \quad x_2 = 4 - 2x_1$

$k \to \infty \qquad x_1 = 0 \quad \text{or} \quad 12 - 3x_1 - 4x_2 = 0 \quad \Rightarrow \quad x_2 = 3 - \frac{3}{4}x_1$

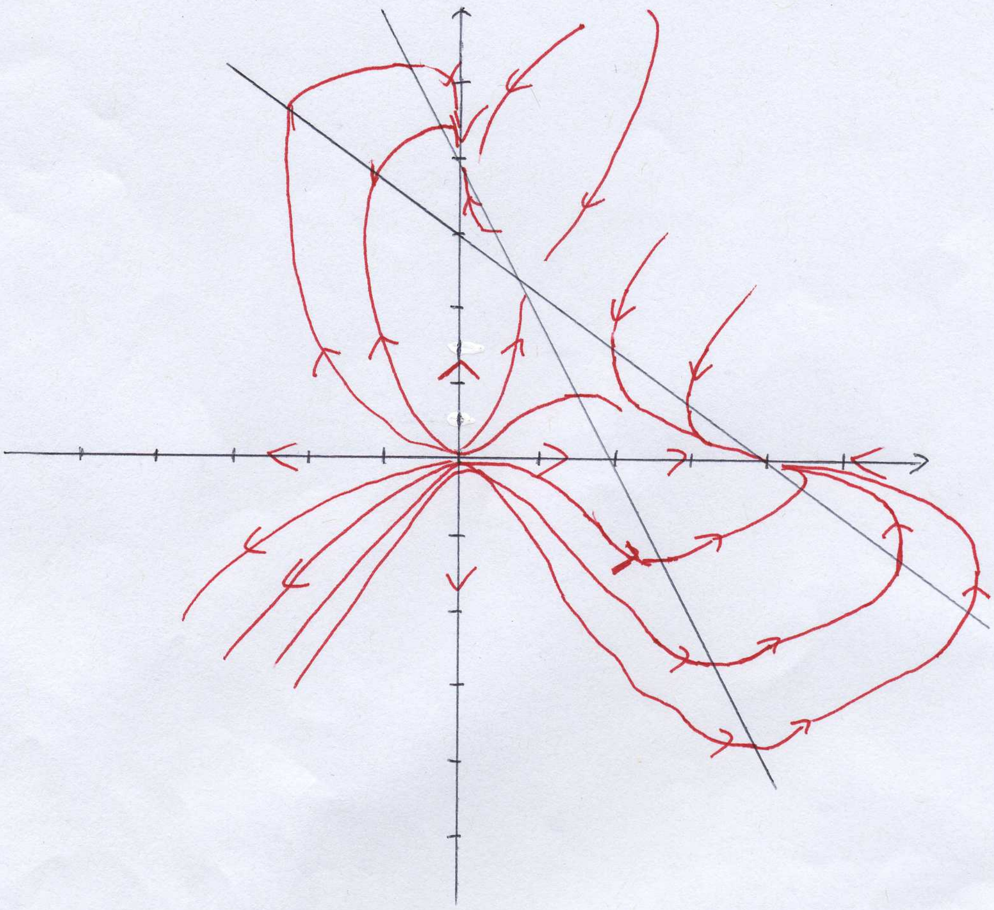


$\dot{x}_2 > 0 \quad x_2 > 0 \wedge x_2 < 4 - 2x_1$

/// or $x_2 < 0 \wedge x_2 > 4 - 2x_1$

$\dot{x}_1 > 0 \quad x_1 > 0 \wedge x_2 < 3 - \frac{3}{4}x_1$

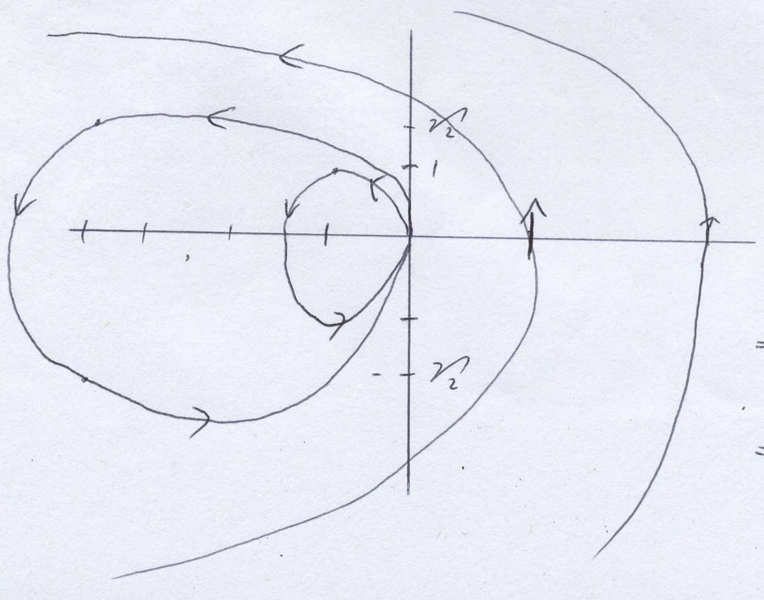
═ or $x_1 < 0 \wedge x_2 > 3 - \frac{3}{4}x_1$



(ii) fixed point: $x_f = (0,0)$

linearisation: $$A(\vec{x}) = \begin{pmatrix} 0 & -7x_2^6 \\ 1 & 4x_2^3 \end{pmatrix}\Bigg|_{x_f} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\frac{\dot{x}_2}{\dot{x}_1} = \frac{x_1 + x_2^4}{-x_2^7} \begin{cases} \to \infty & \text{for } x_2 = 0 \\ \to 0 & \text{for } x_2 = \sqrt[4]{-x_1} \end{cases}$$



det $A = 0$

$\Rightarrow$ nonsimple system

$\Rightarrow$ The linearisation theorem does not apply.