Solution II

1) fixed points:

(1) $x_1(2-x_2)=0 \quad \Rightarrow \quad x_1=0 \quad \text{or} \quad x_2=2$

(2) $x_2(2x_1-x_2-2)=0 \quad \underset{x_1=0}{\Rightarrow} \quad x_2=0 \quad \text{or} \quad x_2=-2$

$\underset{x_2=2}{\Rightarrow} \quad x_1=2$

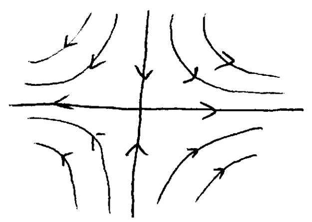
$\Rightarrow \quad \vec{x}_f^{(1)}=(0,0) \quad ; \quad \vec{x}_f^{(2)}=(0,-2) \quad ; \quad \vec{x}_f^{(3)}=(2,2)$

Jacobian matrix:

$$A(\vec{x})=\begin{pmatrix} 2-x_2 & -x_1 \\ 2x_2 & (2x_1-2x_2-2) \end{pmatrix}$$

$\vec{x}_f^{(1)}$: $A(\vec{x}_f^{(1)})=\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \quad \Rightarrow \quad e_1=2 \quad \Rightarrow \quad \vec{v}_1=(1,0)$

$\Rightarrow \quad e_2=-2 \quad \Rightarrow \quad \vec{v}_2=(0,1)$

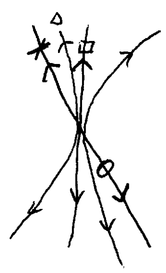


$\equiv$ saddle point ; linearisation theorem works

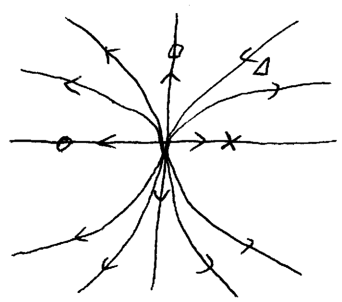
$\vec{x}_f^{(2)}$: $A(\vec{x}_f^{(2)})=\begin{pmatrix} 4 & 0 \\ -4 & 2 \end{pmatrix} \quad \Rightarrow \quad e_1=4 \quad \Rightarrow \quad \vec{v}_1=(-1,2)$

$\Rightarrow \quad e_2=2 \quad \Rightarrow \quad \vec{v}_2=(0,1)$

$J=U^{-1}AU=\begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}$ with $U=(\vec{v}_1,\vec{v}_2)$ unstable node lin. theorem ✓



$\Leftarrow$
$\vec{x}=U\cdot\vec{y}$



$\times \equiv \begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 0 \end{pmatrix}=\begin{pmatrix} -1 \\ 2 \end{pmatrix}$

$o \equiv \begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix}\cdot\begin{pmatrix} -1 \\ 0 \end{pmatrix}=\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$\square \equiv \begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix}\cdot\begin{pmatrix} 0 \\ 1 \end{pmatrix}=\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\Delta \equiv \begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 1 \end{pmatrix}=\begin{pmatrix} -1 \\ 3 \end{pmatrix}$

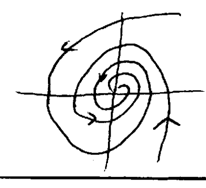
$\vec{x}_f^{(3)}$: $A(\vec{x}_f^{(3)})=\begin{pmatrix} 0 & -2 \\ 4 & -2 \end{pmatrix}$

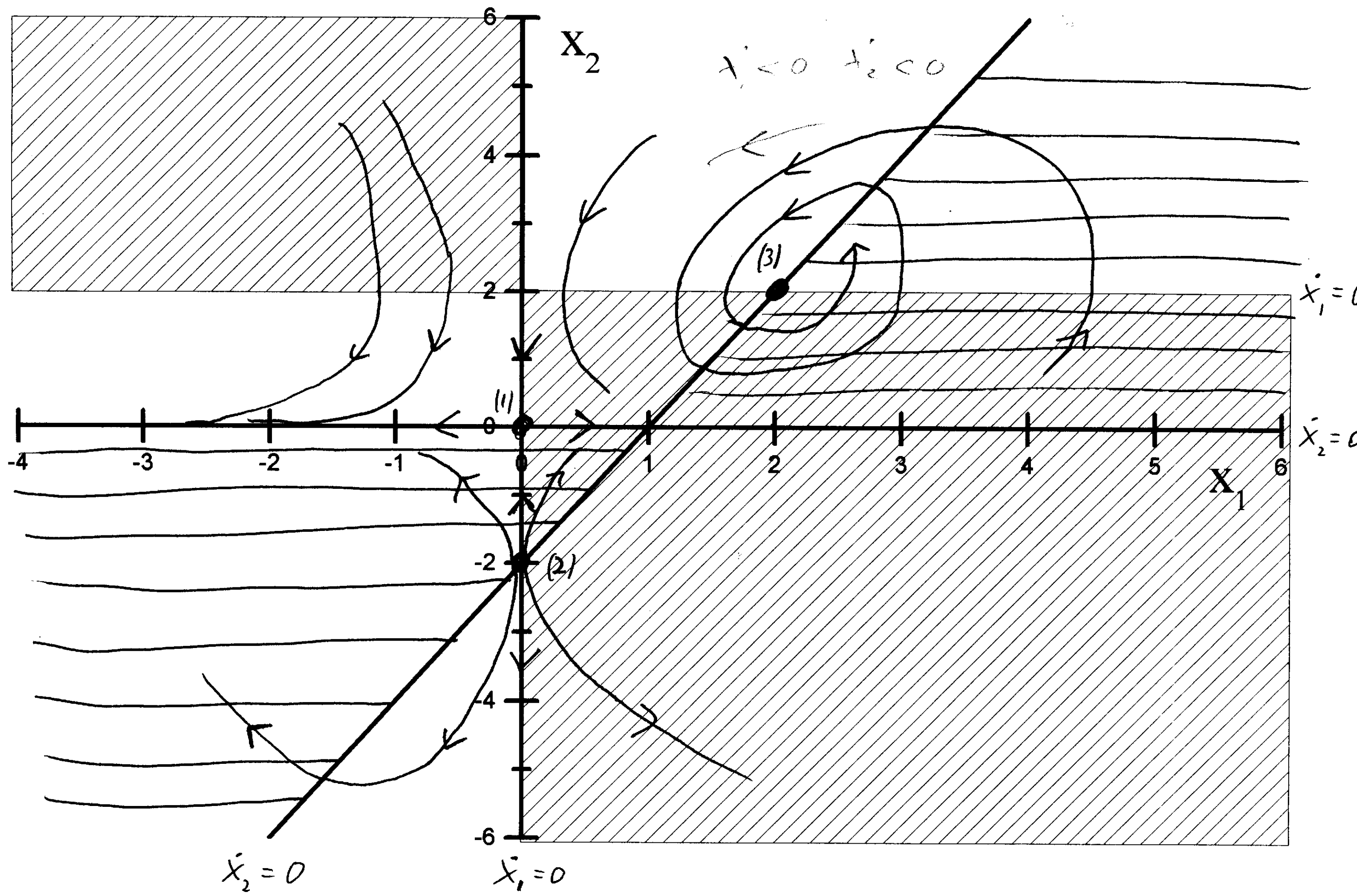
$e_\pm=\pm i\sqrt{7}-1$

$\vec{v}_\pm=\left(\frac{1}{4}(1\pm i\sqrt{7}),1\right)$

stable focus

$J=U^{-1}AU=\begin{pmatrix} -1 & -\sqrt{7} \\ \sqrt{7} & -1 \end{pmatrix}$ with $U=\begin{pmatrix} \sqrt{7}/4 & \frac{1}{4} \\ 0 & 1 \end{pmatrix}$





$x_1(2 - x_2) = 0 \quad \Leftrightarrow \quad \dot{x}_1 = 0 \quad$ for $\quad x_1 = 0 \quad \vee \quad x_2 = 2$

$x_2(2x_1 - x_2 - 2) = 0 \quad \Leftrightarrow \quad \dot{x}_2 = 0 \quad$ for $\quad x_2 = 0 \quad \vee \quad x_2 = 2x_1 - 2$

$\dot{x}_1 > 0 \quad \Leftrightarrow \quad x_1(2 - x_2) > 0 \quad \Leftrightarrow \quad x_1 > 0 \quad \wedge \quad x_2 < 2$

$\qquad\qquad\qquad\qquad\qquad\qquad x_1 < 0 \quad \wedge \quad x_2 > 2 \qquad ///$

$\dot{x}_2 > 0 \quad \Leftrightarrow \quad x_2(2x_1 - x_2 - 2) > 0 \quad \Leftrightarrow \quad x_2 > 0 \quad \wedge \quad x_2 < 2x_1 - 2$

$\qquad\qquad\qquad\qquad\qquad\qquad x_2 < 0 \quad \wedge \quad x_2 > 2x_1 - 2 \qquad \equiv$

2) fixed points:

(1) $\quad x_1(3 - x_1 - x_2) = 0 \qquad x_1 = 0 \quad$ in (2) $\quad x_2 = 0 \quad \vee \quad x_2 = 6$

(2) $\quad x_2(6 - 3x_1 - x_2) = 0 \qquad x_2 = 0 \quad$ in (1) $\quad x_1 = 0 \quad \vee \quad x_1 = 3$

or $\quad 3 - x_1 - x_2 = 0 \quad \wedge \quad 6 - 3x_1 - x_2 = 0 \quad \Rightarrow \quad x_1 = \frac{3}{2} \quad \wedge \quad x_2 = \frac{3}{2}$

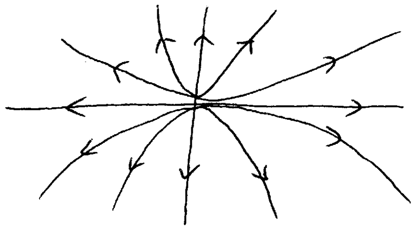
$\Rightarrow \quad \vec{x}_f^{(1)} = (0, 0) \quad ; \quad \vec{x}_f^{(2)} = (3, 0) \quad ; \quad \vec{x}_f^{(3)} = (0, 6) \quad ; \quad \vec{x}_f^{(4)} = \left(\frac{3}{2}, \frac{3}{2}\right)$

- *Jacobian matrix:*

$$A(\vec{x}) = \begin{pmatrix} (3-2x_1-x_2) & -x_1 \\ -3x_2 & (6-3x_1-2x_2) \end{pmatrix}$$

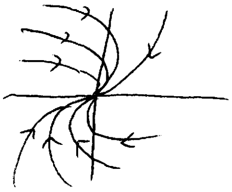
$\vec{x}_f^{(1)}$: $A(\vec{x}_f^{(1)}) = \begin{pmatrix} 3 & 0 \\ 0 & 6 \end{pmatrix} \Rightarrow \begin{matrix} e_1 = 3 & \text{with } \vec{V}_1 = (1,0) \\ e_2 = 6 & \vec{v}_2 = (0,1) \end{matrix}$

$\equiv$ unstable node



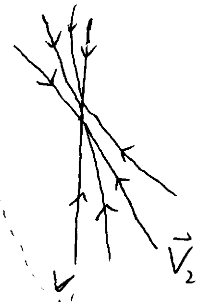
$\vec{x}_f^{(2)}$: $A(\vec{x}_f^{(2)}) = \begin{pmatrix} -3 & -3 \\ 0 & -3 \end{pmatrix} \Rightarrow e_1 = e_2 = -3$ with $\vec{V}_1 = (1,0)$

$J = U^{-1} A U = \begin{pmatrix} -3 & 1 \\ 0 & -3 \end{pmatrix}$ with $U = \begin{pmatrix} 1 & 1 \\ 0 & -\frac{1}{3} \end{pmatrix}$ $\equiv$ stable improper node



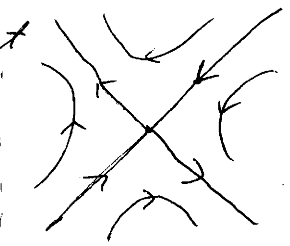
$\vec{x}_f^{(3)}$: $A(\vec{x}_f^{(3)}) = \begin{pmatrix} -3 & 0 \\ -18 & -6 \end{pmatrix} \Rightarrow \begin{matrix} e_1 = -6 & \text{with } \vec{V}_1 = (0,1) \\ e_2 = -3 & \text{with } \vec{V}_2 = (-1,6) \end{matrix}$

$J = U^{-1} A U = \begin{pmatrix} -6 & 0 \\ 0 & -3 \end{pmatrix}$ with $U = (\vec{V}_1, \vec{V}_2)$ $\equiv$ stable node



$\vec{x}_f^{(4)}$: $A(\vec{x}_f^{(4)}) = \begin{pmatrix} -\frac{3}{2} & -\frac{3}{2} \\ -\frac{9}{2} & -\frac{3}{2} \end{pmatrix} \Rightarrow e_\pm = -\frac{3}{2}(1 \pm \sqrt{3})$ with $\vec{V}_\pm = (\pm\frac{1}{\sqrt{3}}, 1)$

$\Rightarrow J = U^{-1} A U = \begin{pmatrix} e_+ & 0 \\ 0 & e_- \end{pmatrix}$ with $U = (\vec{V}_1, \vec{V}_2)$ $\equiv$ saddle point

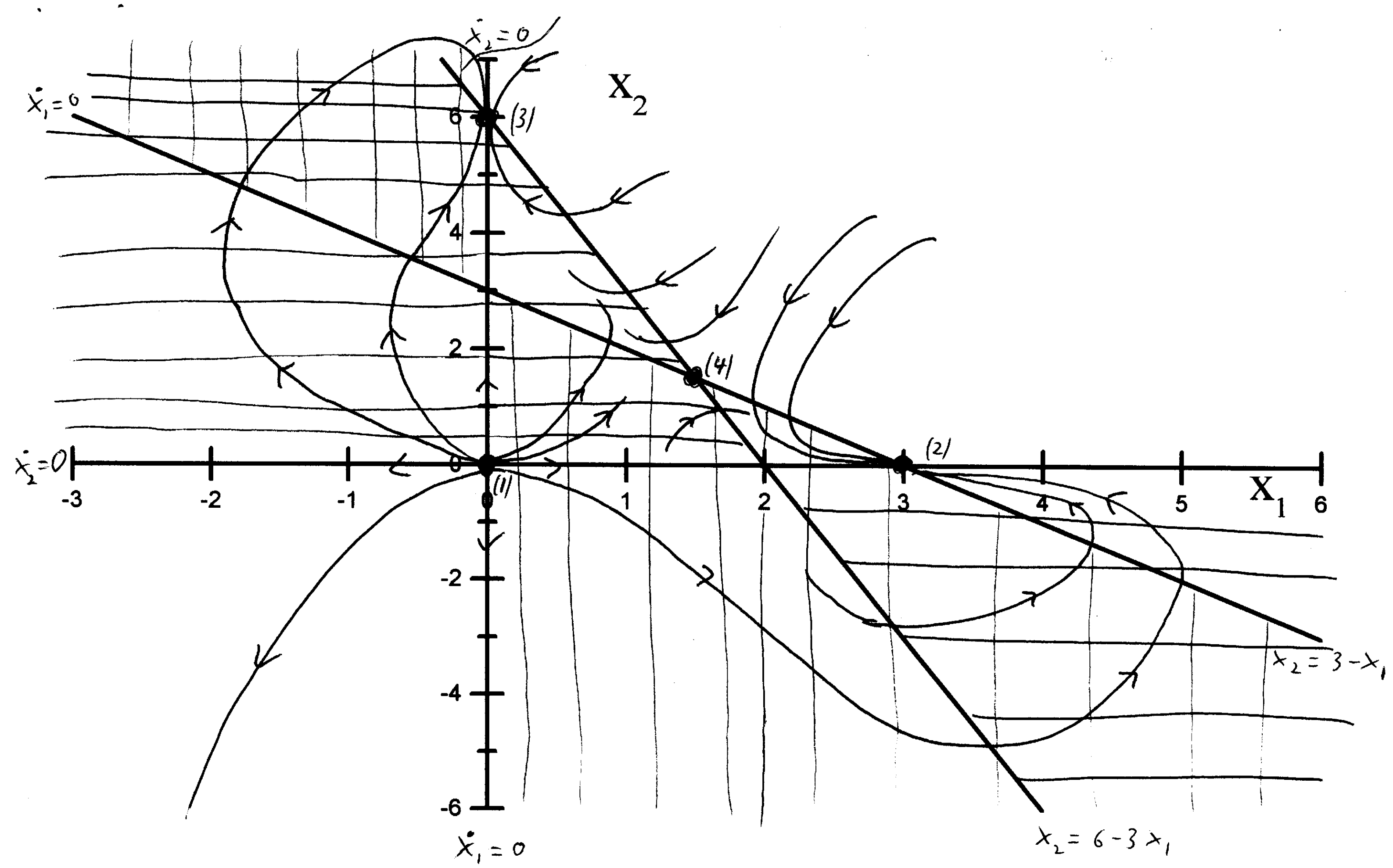


$x_1(3-x_1-x_2) = 0 \iff \dot{x}_1 = 0$ for $x_1 = 0 \lor x_2 = 3 - x_1$

$x_2(6-3x_1-x_2) = 0 \iff \dot{x}_2 = 0$ for $x_2 = 0 \lor x_2 = 6 - 3x_1$

$|||\ \dot{x}_1 > 0 \iff x_1(3-x_1-x_2) > 0 \iff x_1 > 0 \land x_2 < 3-x_1 \lor x_1 < 0 \land x_2 > 3-x_1$

$\equiv \dot{x}_2 > 0 \iff x_2(6-3x_1-x_2) > 0 \iff x_2 > 0 \land x_2 < 6-3x_1 \lor x_2 < 0 \land x_2 > 6-3x_1$



3) *fixed points:*

$-x_1(2 - x_1^2 - x_2^2) = 0$ $\quad$ $x_1 = 0$ into (2) $\quad$ $x_2 = 0$ $\quad$ or $\quad$ $1 + x_2^2 = 0$ } no real sol

$-x_2(1 + x_1^2 + x_2^2 - 3x_1) = 0$ $\quad$ $x_2 = 0$ into (1) $\quad$ $x_1 = 0$ $\quad$ or $\quad$ $2 - x_1^2 = 0 \Rightarrow x_1 = \pm\sqrt{2}$

or $\quad 2 - x_1^2 - x_2^2 = 0 \quad \wedge \quad 1 + x_1^2 + x_2^2 - 3x_1 = 0 \quad \Rightarrow \quad 3 = 3x_1 \quad \Rightarrow x_1 = \cancel{0}\ 1$

$\Rightarrow x_2^2 = 1 \quad x_2 = \pm 1$

$\Rightarrow x_f^{(1)} = (0, 0) \quad ; \quad x_f^{(2)} = (-\sqrt{2}, 0) \quad ; \quad x_f^{(3)} = (\sqrt{2}, 0) \quad ; \quad x_f^{(4)} = (1, -1) \quad ; \quad x_f^{(5)} = (1, 1)$

Jacobian matrix:

$$A(\vec{x}) = \begin{pmatrix} (-2 + 3x_1^2 + x_2^2) & 2x_1x_2 \\ (2x_1x_2 + 3x_2) & (-1 - x_1^2 - 3x_2^2 + 3x_1) \end{pmatrix}$$

$\vec{x}_f^{(1)}$: $\quad A(\vec{x}_f^{(1)}) = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix} \equiv$ stable node

$\vec{x}_f^{(2)}$: $\quad A(\vec{x}_f^{(2)}) = \begin{pmatrix} 4 & 0 \\ 0 & -3-3\sqrt{2} \end{pmatrix} \equiv$ saddle point

$\vec{x}_f^{(3)}$: $\quad A(\vec{x}_f^{(3)}) = \begin{pmatrix} 4 & 0 \\ 0 & -3+3\sqrt{2} \end{pmatrix} \equiv$ unstable node

$x_f^{(4)}$: $\quad A(\vec{x}_f^{(4)}) = \begin{pmatrix} 2 & -2 \\ -1 & -2 \end{pmatrix} \quad \Rightarrow \quad e_\pm = \pm\sqrt{6}$ with $\vec{V}_\pm = (-2 \mp \sqrt{6}, 1)$

$\equiv$ saddle point

$x_f^{(5)}$: $\quad A(x_f^{(5)}) = \begin{pmatrix} 2 & 2 \\ 1 & -2 \end{pmatrix} \quad \Rightarrow \quad e_\pm = \pm\sqrt{6}$ with $\vec{V}_\pm = (2 \pm \sqrt{6}, 1)$

$\equiv$ saddle point

$\vec{x}_f^{(1)}$ is the only stable fixed point, whose neighbourhood we need to study:

We show that $V(\vec{x}) = x_1^2 + x_2^2$ is a strong Lyapunov function for $\vec{x}_f^{(1)}$.

i) ✓ the partial derivatives $\partial V/\partial x_1$, $\partial V/\partial x_2$ exist and are continuous

ii) V is positive definite ✓ (see lecture)

iii) $\frac{dV}{dt}$ is negative definite:

$$\frac{dV}{dt} = \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2$$

$$= 2x_1(-x_1)(2 - x_1^2 - x_2^2) + 2x_2(-x_2)(1 + x_1^2 + x_2^2 - 3x_1)$$

$$= -4x_1^2 + 2x_1^4 + 2\cancel{x_1^2 x_2^2} - 2x_2^2 - 2\cancel{x_1^2 x_2^2} - 2x_2^4 + 6x_1x_2^2$$

$$= -2x_1^2(2 - x_1^2) - 2x_2^2(1 + x_2^2 - 3x_1)$$

$\Rightarrow \dot{V}(\vec{x}_f^{(1)}) = 0 \qquad \dot{V}(\vec{x}) < 0 \quad$ for $\quad x_1^2 < 2 \;\wedge\; 1 + x_2^2 > 3x_1$

$\Rightarrow \dot{V}$ is negative definite for $x_1^2 < 2$ and $x_2^2 > 3x_1 - 1$

$\Rightarrow$ The origin is a stable fixed point. (by Lyapunov stability theorem)

$$x_2^2 > 3x_1 - 1 \quad \text{for} \quad x_1 < \frac{1}{3}$$

$\Rightarrow \quad V(\vec{x}) = x_1^2 + x_2^2 < \frac{1}{3^2}$ is a domain of stability.