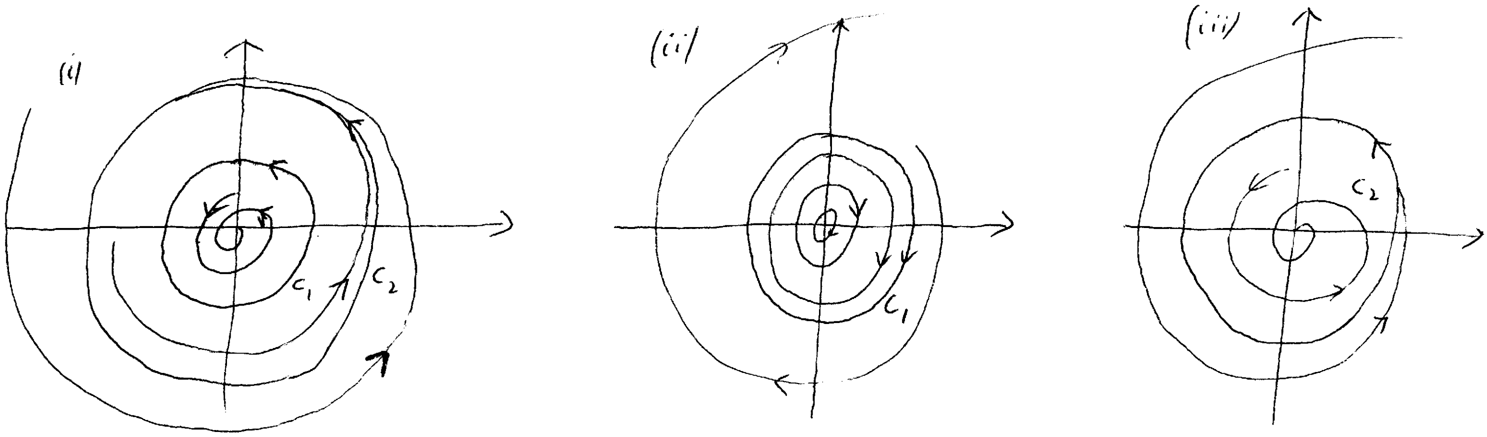
1) i) $\dot r = r(1-r)(r-2)$ $\qquad$ $0<r<1: \ \dot r<0$

$1<r<2: \ \dot r>0$

$\dot\theta = 1$ $\qquad$ $r>2: \ \dot r<0$

$r=0,1,2: \ \dot r = 0$

$$L_\alpha(r) = \begin{cases} r=0 & r=0 \\ C_1 & 0<r<2 \\ C_2 & r=2 \\ \phi & r>2 \end{cases} \qquad L_\omega(r) = \begin{cases} 0 & 0\le r\le 1 \\ C_1 & r=1 \\ C_2 & r>1 \end{cases}$$



ii) $$\dot r = \begin{cases} 0 & r\le 1 \\ r(r-1) & \text{otherwise} \end{cases} \qquad L_\alpha(r) = \begin{cases} C_{r\le 1} & 0\le r\le 1 \\ C_1 & r>1 \end{cases} \qquad L_\omega(r) = \begin{cases} C_{r\le 1} & 0\le r\le 1 \\ \phi & r>1 \end{cases}$$

$\dot\theta = -1$

iii) $\dot r = -r(r-2)^2$ $$L_\alpha(r) = \begin{cases} 0 & r=0 \\ C_2 & r=2 \\ \phi & r>2 \end{cases} \qquad L_\omega(r) = \begin{cases} 0 & r<2 \\ C_2 & r\ge 2 \end{cases}$$

$\dot\theta = 1$

2) for $\vec{x}_f = (0,0)$ $\dot x_1 = \dot x_2 = 0 \Rightarrow \vec{x}_f = (0,0)$ is a fixed point

linearisation: $A = \begin{pmatrix} 2 & -2 \\ 3 & 2 \end{pmatrix}$ eigenvalues $e_\pm = 2 \pm i\sqrt{6}$

- complex eigenvalues with positive real part $\Rightarrow$ type (viii) $\equiv$ unstable focus

i) $x_1 = r\cos\vartheta \quad x_2 = r\sin\vartheta$

$\Rightarrow \quad \dot r\cos\vartheta - r\sin\vartheta\,\dot\vartheta = r\cos\vartheta(2-r^2) - 2r\sin\vartheta \qquad (1)$

$\dot r\sin\vartheta + r\cos\vartheta\,\dot\vartheta = r\sin\vartheta(2-r^2) + 3r\cos\vartheta \qquad (2)$

$(1)\cos\vartheta + (2)\sin\vartheta \Rightarrow \dot{r} = r(2-r^2) + r\sin\vartheta\cos\vartheta$

$(2)\cos\vartheta - (1)\sin\vartheta \Rightarrow r\dot{\vartheta} = r\cos^2\vartheta + 2r \underset{r\neq 0}{\Rightarrow} \dot{\vartheta} = 2 + \cos^2\vartheta$

$\because \dot{\vartheta} \neq 0$ for $r \neq 0 \Rightarrow$ the only fixed point is the origin

ii) for $r=1$: $\dot{r} = 1 + \frac{1}{2}\sin 2\vartheta \geq \frac{1}{2} \quad \forall \vartheta \Rightarrow \dot{r} > 0 \quad \forall \vartheta$

for $r=2$: $\dot{r} = -4 + \sin 2\vartheta \leq -3 \quad \forall \vartheta \Rightarrow \dot{r} < 0 \quad \forall \vartheta$

$\Rightarrow$ trajectories which enter this annular region never leave it

$\because$ there is no fixed point in this region (see (i)), there is at least one limit cycle in the region by the Poincaré - Bendixson theorem

iii) smaller regime:

$\dot{r} > 0 \quad \forall \vartheta \iff r(2-r^2) + r\frac{1}{2}\sin 2\vartheta > 0$

$\iff r^2 < \frac{1}{2}\sin 2\vartheta + 2$ inner boundary

$r^2 \leq \min\left(\frac{1}{2}\sin 2\vartheta + 2\right)$

$r \leq \sqrt{3/2}$

$\dot{r} < 0 \quad \forall \vartheta \iff r^2 > \frac{1}{2}\sin 2\vartheta + 2$ outer boundary

$r^2 \geq \max\left(\frac{1}{2}\sin 2\vartheta + 2\right)$

$r \geq \sqrt{5/2}$

$\therefore$ In $D^\varepsilon = \{(r,\vartheta) : \sqrt{3/2} - \varepsilon \leq r \leq \sqrt{5/2} + \varepsilon\}$ there is no fixed point

$\because \dot{r} > 0$ on the inner boundary and $\dot{r} < 0$ on the outer boundary, the trajectories which enter D^ε do not leave it

$\therefore$ by the PBT $\exists$ at least one limit cycle in D^ε

$\because r = \sqrt{3/2}$ and $r = \sqrt{5/2}$ are not trajectories of the system the former conclusion also holds for $\varepsilon = 0$

$\therefore$ In the annular region $D = \{(r,\vartheta) : \sqrt{3/2} \leq r \leq \sqrt{5/2}\}$ the system has a periodic orbit

3) i) $\dot{x}_1 = x_1 + 3x_2^2 \qquad \dot{x}_2 = -2x_1 - x_2(1 + x_1^2 + x_1^4)$

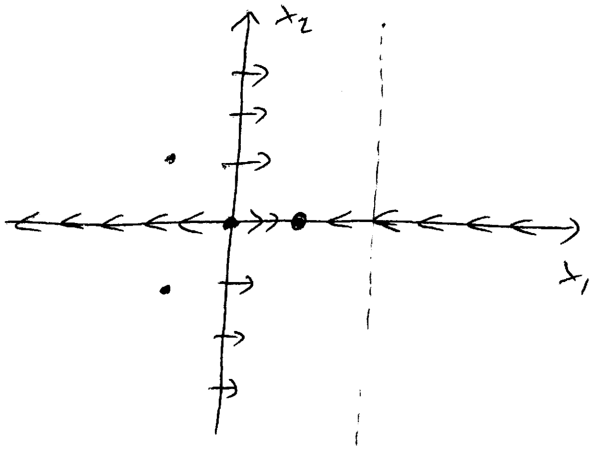
$$\operatorname{div} X = 1 - 1 - x_1^2 - x_1^4 = -(x_1^2 + x_1^4) < 0$$

$\Rightarrow$ there is no limit cycle in the entire plane by the Bendixson criter

ii) $\dot{x}_1 = x_1 - x_1^2 + 2x_2^2 \qquad \dot{x}_2 = x_1 x_2 + x_2$

$$\operatorname{div} X = 1 - 2x_1 + x_1 + 1 = 2 - x_1$$

$\Rightarrow$ by the BC there is no limit cycle for $x_1 > 2$ and $x_2 < 2$



$\left.\begin{array}{l}\dot{x}_2 = 0 \\ \dot{x}_1 = x_1(1 - x_1)\end{array}\right\}$ for $x_2 = 0$

$\left.\begin{array}{l}\dot{x}_1 = 2x_2^2 \\ \dot{x}_2 = x_2\end{array}\right\}$ for $x_1 = 0$

four fixed points: $(0,0) = x_f^{(1)}$, $x_f^{(2)} = (1,0)$, $x_f^{(3)} = (-1,1)$, $x_f^{(4)} = (-1,-1$

$\Rightarrow$ We can not draw any trajectory which encircles a fixed point and crosses the line $x_1 = 2$. Therefore it follows by the BC and Theorem 5 that there is no limit cycle in the entire plane.