Dynamical Systems II

Coursework 1

Hand in the complete solutions to all three questions in the general office (room C123). Each question carries 20 marks.

DEADLINE: Tuesday 11/11/2008 at 15:00

---

1) A competing species model is given by the following dynamical system

$$\begin{aligned} \dot{x}_1 &= x_1(\gamma_1 - s_1 x_1 - \varepsilon_1 x_2) \\ \dot{x}_2 &= x_2(\gamma_2 - s_2 x_2 - \varepsilon_2 x_1) \end{aligned} \qquad \text{with} \quad \gamma_i, s_i, \varepsilon_i \in \mathbb{R}^+, i = 1, 2,$$

where x_i is the population of the species i, γ_i denotes the growth rate of the species i, and γ_i/s_i is the saturation level for the population of the species i.

i) Explain briefly the meaning of the parameters ε_i. Which scenario is described when $\varepsilon_1 = \varepsilon_2 = 0$?

ii) Take from now on the parameters of the model to be fixed as

$$\gamma_1 = s_1 = s_2 = \varepsilon_1 = 1, \qquad \varepsilon_2 = 2, \qquad \text{and} \qquad \gamma_2 = \frac{3}{2}.$$

Using these values determine all fixed points for the competing species model.

iii) Decide for each of the fixed points whether it is possible for the two species to coexist.

iv) State the linearization theorem and decide whether it can be applied at each of the fixed points. Determine the nature of the fixed points.

v) Compute the isoclines for the model and use this information to sketch the phase portrait. Indicate the regions in which $\dot{x}_1 > 0$ and $\dot{x}_2 > 0$. Precise local phase portraits at each fixed point are not required.

2) Consider the dynamical system of the form

$$\begin{aligned} \dot{x}_1 &= -3x_1 - \frac{1}{2}x_2^3 + x_1 x_2^2 \\ \dot{x}_2 &= 2x_1 x_2^2 + 2x_2 x_1^2 \end{aligned}$$

i) State the Lyapunov stability theorem and the definitions for a weak and strong Lyapunov function. Show that the function

$$V(x_1, x_2) = 8x_1^2 + 2x_2^2$$

is a weak Lyapunov function for the system specified above. Deduce from this the stability properties of the fixed point.

ii) Determine the length of the major and minor axis of the ellipse which confines the maximal domain of stability.

iii) State the extension of the Lyapunov stability theorem and conclude from it that the fixed point is asymptotically stable.

3) Consider the dynamical system of the form

$$\begin{aligned} \dot{x}_1 &= 25x_1 - x_2 - x_1(x_1^2 + x_2^2) \\ \dot{x}_2 &= x_1 + 25x_2 - x_2(x_1^2 + x_2^2) \ . \end{aligned}$$

i) Change the variables of the system to polar coordinates, using the conventions $x_1 = r\cos\vartheta$ and $x_2 = r\sin\vartheta$. Deduce from the equation for $\dot{\vartheta}$ that the origin is the only fixed point of the system.

ii) State the Poincaré-Bendixson theorem. Employ the theorem to conclude that the system has at least one limit cycle in the annular region

$$\mathcal{D} = \{(r, \vartheta) : 4 \le r \le 6\} \ .$$

iii) Sketch the phase portrait for the above system. Determine exactly the limit cycle in $\mathcal{D}$.

iv) State the definitions for the α-limit set and the ω-limit set and determine them thereafter. Decide whether the limit cycle in *iii)* is stable, unstable or semistable.

v) State a sufficient condition for the non-existence of a limit cycle in a particular domain.