Dynamical Systems II

Coursework 1

Hand in the complete solutions to all three questions in the general office (room C123).

DEADLINE: Tuesday 10/11/2009 at 16:00

1) Consider the linear dynamical system of the form [20]

$$\begin{aligned}\dot{x}_1 &= ax_1 + bx_2 \\ \dot{x}_2 &= cx_1 + dx_2\end{aligned} \qquad \text{with} \quad a, b, c, d \in \mathbb{R}.$$

Assume that the Jacobian matrix of this system has the two eigenvalues $\alpha \pm i\beta$, with $\alpha, \beta \in \mathbb{R}$ with $\alpha < 0$. Prove that the origin is a stable focus.

2) Consider the dynamical system of the form [20]

$$\begin{aligned}\dot{x}_1 &= x_2 + x_1(3 - x_1^2 - x_2^2)(x_1^4 + x_2^4 + 2x_1^2 x_2^2 - 1)^2 \\ \dot{x}_2 &= -x_1 + x_2(3 - x_1^2 - x_2^2)(x_1^4 + x_2^4 + 2x_1^2 x_2^2 - 1)^2\end{aligned}$$

i) Change the variables of the system to polar coordinates, using the conventions $x_1 = r\cos\vartheta$ and $x_2 = r\sin\vartheta$. Deduce that the origin is the only fixed point of the system.

ii) Use the Poincaré-Bendixson theorem to argue that there is at least one limit cycle in the annular region

$$\mathcal{D} = \{(r, \vartheta) : 1/2 \leq r \leq 2\} \ .$$

iii) Sketch the phase portrait for the above system.

iv) Determine all α and ω limit sets. Compute all limit cycles explicitly and decide whether they are stable, unstable or semistable. Compare your results with your answer to *ii)*. Can you use the Poincaré-Bendixson theorem to optimize the domain $\mathcal{D}$ to compute all limit cycles?

v) Can you use Bendixson's criterium to argue that there can not be any limit cycle in the domains ($\varepsilon \ll 1$)

$$\hat{\mathcal{D}} = \{(r, \vartheta) : 3 \leq r \leq 5\} \ ?$$

3) Consider the dynamical system of the form [10]

$$\dot{x}_1 = x_1 x_2^2 - 9x_1 - 16x_2^3 \qquad \dot{x}_2 = 4x_1 x_2^2 + 2x_2 x_1^2.$$

i) Show that the function

$$V(x_1, x_2) = 4x_1^2 + 16x_2^2$$

is a weak Lyapunov function for the dynamical system specified above. Deduce from this the stability properties of the fixed point.

ii) Determine the length of the major and minor axis of the ellipse which confines the maximal domain of stability.

iii) State the extension of the Lyapunov stability theorem and conclude from it that the fixed point is asymptotically stable.