Dynamical Systems II

Coursework 1 (Solutions)

Hand in the complete solutions to all three questions in the general office (room C123).

DEADLINE: Tuesday 10/11/2009 at 16:00

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1) Consider the linear dynamical system of the form

$$\begin{aligned} \dot{x}_1 &= ax_1 + bx_2 \\ \dot{x}_2 &= cx_1 + dx_2 \end{aligned} \qquad \text{with} \quad a, b, c, d \in \mathbb{R}. \tag{1}$$

The Jacobian is

$$J = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The eigenvalues are computed from $\det(J - \lambda\mathbb{I}) = 0$, i.e.

$$\lambda^2 - (a+d)\lambda + ad - bc = 0,$$

which is solved by

$$\lambda_\pm = \frac{a+d}{2} \pm \frac{1}{2}\sqrt{a^2 + 4bc - 2ad + d^2} = \alpha \pm i\beta.$$

This means

$$a^2 + 4bc - 2ad + d^2 = (a-d)^2 + 4bc < 0. \tag{2}$$

Let us now assume that the equations (1) have a solution of the general form

$$x_1 = Ae^{\lambda t}, \qquad x_2 = Be^{\lambda t} \qquad \text{for } A, B \in \mathbb{R}. \tag{3}$$

Then, substituting this into (1) gives

$$\begin{aligned} A\lambda e^{\lambda t} &= aAe^{\lambda t} + bBe^{\lambda t}, \\ B\lambda e^{\lambda t} &= cAe^{\lambda t} + dBe^{\lambda t}, \end{aligned}$$

which corresponds to

$$\begin{aligned} (a-\lambda)A + bB = 0, \\ cA + (d-\lambda)B = 0, \end{aligned} \qquad \Leftrightarrow \qquad (J - \lambda\mathbb{I})\begin{pmatrix} A \\ B \end{pmatrix} = 0.$$

These equations only have nontrivial solutions when $\det(J - \lambda\mathbb{I}) = 0$, such that the general solution (3) becomes

$$
\begin{aligned}
x_1 &= A_1 e^{\lambda_+ t} + A_2 e^{\lambda_- t} = e^{\alpha t}(A_1 e^{i\beta t} + A_2 e^{i\beta t}), \\
x_2 &= B_1 e^{\lambda_+ t} + B_2 e^{\lambda_- t} = e^{\alpha t}(B_1 e^{-i\beta t} + B_2 e^{-i\beta t}).
\end{aligned}
$$

Clearly for $\alpha < 0$ we have $x_{1/2} \to 0$ for $t \to \infty$.

Next we establish that the trajectory spirals into the origin. Using $x_1 = r\cos\vartheta$ and $x_2 = r\sin\vartheta$ we have $\vartheta = \arctan x_2/x_1$, such that

$$
\begin{aligned}
\dot{\vartheta} &= \frac{\partial\vartheta}{dx_1}\dot{x}_1 + \frac{\partial\vartheta}{dx_2}\dot{x}_2 = -\frac{x_2\dot{x}_1}{x_1^2 + x_2^2} + \frac{x_1\dot{x}_2}{x_1^2 + x_2^2} \\
&= \frac{x_1(cx_1 + dx_2) - x_2(ax_1 + bx_2)}{x_1^2 + x_2^2} = \frac{cx_1^2 - bx_2^2 + (d-a)x_1x_2}{x_1^2 + x_2^2}.
\end{aligned}
$$

We have used here $\partial\vartheta/dx_1 = -x_2/(x_1^2 + x_2^2)$ and $\partial\vartheta/dx_2 = x_1/(x_1^2 + x_2^2)$.

From (2) we know that b and c must have opposite signs.

- for $x_2 = 0$ we have $\dot{\vartheta} = c > 0$ for $b < 0$, $c > 0$ or $\dot{\vartheta} = c < 0$ for $b > 0$, $c < 0$
- for $x_2 \neq 0$ we have $\dot{\vartheta} \neq 0$ or $cx_1^2 - bx_2^2 + (d-a)x_1x_2 = 0$. From the latter follows

$$
c\frac{x_1^2}{x_2^2} - b + (d-a)\frac{x_1}{x_2} = 0,
$$

which means

$$
\frac{x_1}{x_2} = \frac{a-d}{2c} \pm \frac{1}{2c}\sqrt{a^2 + 4bc - 2ad + d^2}. \tag{4}
$$

As we know from (3) that $a^2 + 4bc - 2ad + d^2 < 0$ there is no real solution to (4) and therefore $\dot{\vartheta} \neq 0$ for $x_2 \neq 0$.

Therefore when $b < 0$, $c > 0$, $a + d < 0$ the trajectories spiral anti-clockwise into the origin and when $b > 0$, $c < 0$, $a + d < 0$ the trajectories spiral clockwise into the origin. $\boxed{\sum = 20}$

2) We have

$$
\begin{aligned}
\dot{x}_1 &= x_2 + x_1(3 - x_1^2 - x_2^2)(x_1^4 + x_2^4 + 2x_1^2x_2^2 - 1)^2 \\
\dot{x}_2 &= -x_1 + x_2(3 - x_1^2 - x_2^2)(x_1^4 + x_2^4 + 2x_1^2x_2^2 - 1)^2
\end{aligned}
$$

i) With $x_1 = r\cos\vartheta$ and $x_2 = r\sin\vartheta$, we obtain

$$
\begin{aligned}
\dot{x}_1 &= \dot{r}\cos\vartheta - r\sin\vartheta\dot{\vartheta} = r\sin\vartheta + r\cos\vartheta(3 - r^2)(r^4 - 1)^2 && (5) \\
\dot{x}_2 &= \dot{r}\sin\vartheta + r\cos\vartheta\dot{\vartheta} = -r\cos\vartheta + r\sin\vartheta(3 - r^2)(r^4 - 1)^2 && (6)
\end{aligned}
$$

where we used $x_1^2 + x_2^2 = r^2$ and $x_1^4 + x_2^4 + 2x_1^2x_2^2 = r^4$.
Computing (5)$\cos\vartheta$+(6)$\sin\vartheta$ gives

$$
\dot{r} = r(3 - r^2)(r^4 - 1)^2 \tag{7}
$$

and (6)cos ϑ-(5)sin ϑ yields

$$r\dot{\vartheta} = -r \quad \Rightarrow \quad \dot{\vartheta} = -1. \tag{8}$$

Since $\dot{\vartheta} \neq 0$ for all time, the origin is the only fixed point. [5]

ii) With (7) we compute $\dot{r}$ at the inner and outer boundary of

$$\mathcal{D} = \{(r, \vartheta) : 1/2 \leq r \leq 2\} \ .$$

We find

$$\dot{r}(r = 1/2) = \frac{1}{2}\frac{11}{4}\left(-\frac{15}{16}\right)^2 > 0$$
$$\dot{r}(r = 2) = 2(-1)15^2 < 0.$$

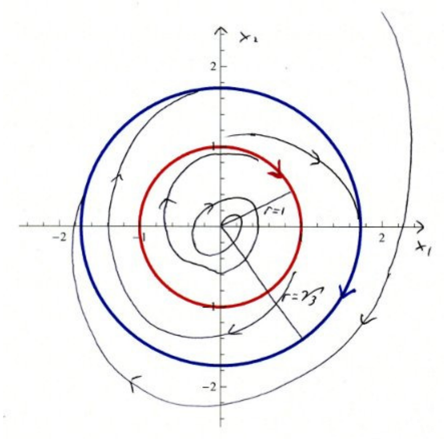
$\Rightarrow$ trajectories which enter $\mathcal{D}$ never leave it.

$\Rightarrow$ It follows by the Poincaré-Bendixson theorem that there is at least one limit cycle in $\mathcal{D}$. [3]

iii) We have

$$\begin{array}{ll} \dot{r} = 0 & \text{for } r = 0, 1, \sqrt{3} \ \equiv \ \text{limit cycles} \\ \dot{r} > 0 & \text{for } 0 < r < 1 \\ \dot{r} > 0 & \text{for } 1 < r < \sqrt{3} \\ \dot{r} < 0 & \text{for } \sqrt{3} < r \\ \dot{\vartheta} = -1 & \equiv \ \text{clockwise direction} \end{array}$$

Assembling all this gives the following phase portrait. [5]



iv) The α and ω limit sets are

$$L_\alpha(\vec{x}) = \begin{cases} (0,0) & \text{for } \ r < 1 \\ C_1 & \text{for } \ 1 \leq r < \sqrt{3} \\ C_{\sqrt{3}} & \text{for } \ r = \sqrt{3} \\ \varnothing & \text{for } \ \sqrt{3} < r \end{cases} \qquad L_\omega(\vec{x}) = \begin{cases} (0,0) & \text{for } \ r = 0 \\ C_1 & \text{for } \ 0 < r \leq 1 \\ C_{\sqrt{3}} & \text{for } \ 1 < r \end{cases}$$

• C_1 is semistable and $C_{\sqrt{3}}$ is stable.

• These findings are consistent with the conclusions drawn in *ii*), both limit cycles are in $\mathcal{D}$.

• Defining a domain

$$\mathcal{D}' = \left\{(r,\vartheta) : \sqrt{3}-\varepsilon \leq r \leq \sqrt{3}+\varepsilon\right\}$$

we can use the same arguments as in *ii*) to conclude that there is a limit cycle in $\mathcal{D}'$.

• However, we can not define such an annular region around C_1, as for

$$\mathcal{D}'' = \{(r,\vartheta) : 1-\varepsilon \leq r \leq 1+\varepsilon\}$$

the trajectories will enter $\mathcal{D}''$ on the inner boundary and leave it at the outer boundary. Thus the Poincaré-Bendixson theorem can not be applied in this case. 6

v) Bendixson's criterium requires the region to be simply connected, which is not the case for $\hat{\mathcal{D}}$. Therefore the criterium can not be applied. 1

$\sum = 20$

3) We have

$$\begin{aligned}\dot{x}_1 &= x_1 x_2^2 - 9x_1 - 16x_2^3 \\ \dot{x}_2 &= 4x_1 x_2^2 + 2x_2 x_1^2\end{aligned}$$

i) **Lyapunov stability theorem:** *Consider the system $\dot{\vec{x}} = \vec{X}(\vec{x})$ with a fixed point at the origin. If there exists a real valued function $V(\vec{x})$ in a neighbourhood $N(\vec{x}=0)$ such that:*

a) *the partial derivatives $\partial V/\partial x_1$, $\partial V/\partial x_2$ exist and are continuous*

b) *the function $V(\vec{x})$ is positive definite*

c) *dV/dt is negative semi-definite (definite)*

then the origin is a stable (asymptotically stable) fixed point.

Def.: *A function V for which the conditions i)-iii) hold with iii) semi-definite is called weak Lyapunov function.*

Def.: *A function V for which the conditions i)-iii) hold with iii) definite is called strong Lyapunov function.*

Verify the requirements a)-c) for $V(x_1, x_2) = 4x_1^2 + 16x_2^2$

a) Clearly the partial derivatives $\partial V/\partial x_1$, $\partial V/\partial x_2$ exist and are continuous.

b) $\because V(0,0) = 0$ and $V(x_1,x_2) > 0$ for $(x_1,x_2) \neq 0$
$\Rightarrow$ the function $V(x_1,x_2) = 4x_1^2 + 16x_2^2$ is positive definite

c) Compute dV/dt:

$$\begin{aligned}\dot{V} &= \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2 \\ &= 8x_1(x_1x_2^2 - 9x_1 - 16x_2^3) + 32x_2(4x_1x_2^2 + 2x_2x_1^2) \\ &= 8x_1^2x_2^2 - 72x_1^2 \underline{-128x_1x_2^3} + \underline{128x_1x_2^3} + 64x_2^2x_1^2 \\ &= -8x_1^2[9 - x_2^2(1+8)] \\ &= -72x_1^2(1-x_2^2)\end{aligned}$$

$\Rightarrow \dot{V} = 0$ for $(0, x_2)$
$\Rightarrow \dot{V} \leq 0$ for $|x_2| < 1$
$\Rightarrow$ is negative semi-definite

$\Rightarrow V$ is a weak Lyapunov function
$\Rightarrow$ by the Lyapunov stability theorem follows that the origin is a stable fixed point. [6]

ii) For $|x_2| < 1$ all points inside the level curve of $V(x_1, x_2) = 4x_1^2 + 16x_2^2$ will be dragged to the origin.

$$\Rightarrow 4x_1^2 + 16x_2^2 < 16 \quad \Rightarrow \frac{x_1^2}{4} + x_2^2 < 1$$

$\Rightarrow$ the length of the minor is 2 and the length of the major is 4. [2]

iii) **Corollary:** *Let $V(\vec{x}(t))$ be a weak Lyapunov function for the system $\dot{\vec{x}} = \vec{X}(\vec{x})$ in a neighbourhood of an isolated fixed point $\vec{x}_f = (0,0)$. Then if $\dot{V} \neq 0$ on a trajectory, except for the fixed point, the origin is asymptotically stable.*

For $(0, x_2)$ the dynamical system reduces to $\dot{x}_1 = -16x_2^3$ and $\dot{x}_2 = 0$. [2]

This means the line $(0, x_2)$ is not a trajectory and therefore it follows from the corollary that the origin is asymptotically stable. [$\sum = 10$]