Dynamical Systems II

Coursework 1

Hand in the complete solutions to both questions in the SEMS general office (C109).

DEADLINE: Thursday 11/11/2010 at 16:00

1) Consider the following second order differential equation

$$\ddot{x} + \dot{x} + \mu x^3 + \nu \dot{x}^5 = 0 \qquad \mu \in \mathbb{R}^+, \nu \in \mathbb{R}.$$

(i) Find a suitable variable transformation which changes the second order differential equation into a system of two first order differential equations.

(ii) Determine the fixed point of the system. State the linearization theorem and decide whether it can be used to draw conclusions from it with regard to the stability of the fixed point.

(iii) State the Lyapunov stability theorem. Take the function

$$V(x_1, x_2) = \alpha x_1^4 + 2x_2^2, \qquad \alpha \in \mathbb{R}^+$$

as a candidate for a Lyapunov function. Find a relation between α and μ and a range for ν, such that $V(x_1, x_2)$ becomes a weak Lyapunov function for the dynamical system constructed in (i).

(iv) For $\alpha = \mu$ and $\nu = 0$ decide whether any of the two functions

$$V_1(x_1, x_2) = x_1^2 + x_2^2 \qquad \text{and} \qquad V_2(x_1, x_2) = x_1^3 + x_2^3$$

is a Lyapunov function for the above system.

(v) State the extension of the Lyapunov stability theorem and conclude from it that the fixed point is asymptotically stable for the values of α, μ and ν found in (iii).

2) Consider the dynamical system of the form

$$\begin{aligned} \dot{x}_1 &= x_2 + x_1(x_1^2 + x_2^2 - 5)(1 - x_1^4 - x_2^4 - 2x_1^2 x_2^2) \\ \dot{x}_2 &= -x_1 + x_2(x_1^2 + x_2^2 - 5)(1 - x_1^4 - x_2^4 - 2x_1^2 x_2^2) \end{aligned}$$

(i) Change the variables of the system to polar coordinates, using the conventions $x_1 = r\cos\vartheta$ and $x_2 = r\sin\vartheta$. Deduce that the origin is the only fixed point of the system.

(ii) Use the Poincaré-Bendixson theorem to argue that there is at least one limit cycle in the annular region

$$\mathcal{D} = \{(r,\vartheta) : 2 \leq r \leq 3\} \ .$$

(iii) Sketch the phase portrait for the above system.

(iv) State the defininiton for α and ω limit sets and subsequently compute all of them. State the definition of a limit cycle using the notion of α and ω limit sets. Compute the limit cycles explicitly and decide whether they are stable, unstable or semistable. Compare your results with your answer to (ii).

(v) State Bendixson's criterium and decide whether it can be used to argue that the domain

$$\hat{\mathcal{D}} = \{(r,\vartheta) : 3 \leq r \leq 5\}$$

contains a limit cycle.