Dynamical Systems II

Coursework 2

Hand in the complete solutions to all three questions in the general office (room C123). Each question carries 20 marks.

DEADLINE: Tuesday 09/12/2008 at 16:00

1) *i)* Consider the dynamical system

$$\dot{x}_1 = 7x_2$$
$$\dot{x}_2 = -(x_1^2 - \lambda)x_2 - 7x_1 - 2x_1^3$$

with $\lambda \in \mathbb{R}$ being a bifurcation parameter. Use the stability index

$$I = \omega\left(Y^1_{111} + Y^1_{122} + Y^2_{112} + Y^2_{222}\right) + Y^1_{11}Y^2_{11} - Y^1_{11}Y^1_{12} + Y^2_{11}Y^2_{12}$$
$$+Y^2_{22}Y^2_{12} - Y^1_{22}Y^1_{12} - Y^1_{22}Y^2_{22}$$

to argue that the origin is asymptotically stable for $\lambda = 0$. As in the lecture we abbreviated $Y^i_{jk} = \partial^2 Y_i/\partial y_j \partial y_k$ and $Y^i_{jkl} = \partial^2 Y_i/\partial y_j \partial y_k \partial y_l$.

ii) Prove that the system possesses a Hopf bifurcation for $\lambda = 0$.

iii) For the following system

$$\dot{r} = \lambda r(r - 2\lambda)^2 \qquad \text{and} \qquad \dot{\vartheta} = -1$$

sketch the phase portrait for positive λ and determine the α and ω limit sets. Sketch the bifurcation diagram in the (r, λ)-plane with $\lambda \in \mathbb{R}$ being the bifurcation parameter. Decide which type of bifurcation occurs at the point $(r, \lambda) = (0, 0)$.

2) *i)* Exploit the fact that the following system

$$H(x_1, x_2) = \frac{1}{2}x_2^2 + g e^{-x_1}\cos x_1 \qquad g \in \mathbb{R}$$

is a potential system to find and classify all its fixed points.

ii) Taking the constant $g = 1/2$, determine the separatices for the system and draw a phase portrait for $x_1 \in [-5, 5]$.

iii) Given the initial condition $x_1 = 0$, $x_2 = 1/2$ compute the period T for the potential system

$$H(x_1, x_2) = \frac{1}{2}x_2^2 + \frac{1}{8}x_1^8.$$

Hint: You may use the integral $\int_0^1 dx/\sqrt{1 - x^8} = \sqrt{\pi}\,\Gamma(9/8)/\Gamma(5/8)$.

3) Consider the following difference equation

$$x_{n+1} = F(x_n) = \lambda x_n(4 - x_n) \qquad \text{for } \lambda \in \mathbb{R}^+.$$

λ is taken to be the bifurcation parameter.

i) Depending on the values of λ, determine the nature of the fixed points and their stability.

ii) State the condition which determines the existence of a 2-cycle. Show that 2-cycles for the above system are determined by the solutions of the equation

$$x^2\lambda^2 - 4x\lambda^2 - x\lambda + 4\lambda + 1 = 0 \ .$$

Compute the solution of this equation and use it to argue that the existence of a 2-cycle requires $\lambda \geq 3/4$.

iii) Determine the domain of stability for the 2-cycle and sketch the corresponding bifurcation diagram.