Dynamical Systems II

Coursework 2

Hand in the complete solutions to all four questions in the general office (room C123).

DEADLINE: Thursday 10/12/2009 at 14:00

1) Consider the one dimensional dynamical system of the form [10 marks]

$$\dot{x} = x^3 + \alpha x^2 + \beta x \quad \text{with } \alpha, \beta \in \mathbb{R}.$$

Find the fixed points for the system. For $\alpha = 0$ find the pitchfork bifurcation point for the above system. Show that when $\alpha \neq 0$ this point transforms into a transcritical bifurcation point. Find the turning point of the system as a function of α.

2) *i)* Prove that the van der Pol differential equation [15 marks]

$$\ddot{x} + \lambda(x^2 - 1)\dot{x} + x = 0$$

possesses a Hopf bifurcation.

ii) For the initial conditions $x = 0$ and $\dot{x} = 1$ choose two values of λ which lead to qualitatively different behaviour. Produce sketches for the corresponding phase portraits by using the wolfram alpha web site:
http://www.wolframalpha.com/input/?i=dynamical+systems=.
(The link is also available from the course web site.)

3) *i)* Use the fact that the following system is a potential system [15 marks]

$$H(x_1, x_2) = \frac{1}{2}x_2^2 + \kappa e^{-x_1} \sin x_1 \qquad \kappa \in \mathbb{R}$$

to find and classify all its fixed points.

ii) Sketch the phase portrait.

iii) Given the initial condition $x_1 = 2^{3/4}$, $x_2 = 2$ compute the period T for the potential system

$$H(x_1, x_2) = \frac{1}{2}x_2^2 + \frac{1}{4}x_1^4.$$

Hint: You may use the integral $\int_0^1 dx/\sqrt{1 - x^4} = \sqrt{\pi}\Gamma(5/4)/\Gamma(3/4)$.

4) Consider the harmonic oscillator in two dimensions described by the Hamiltonian [10 marks]

$$H(x_1, x_2, p_1, p_2) = \frac{1}{2m}(p_1^2 + p_2^2) + \frac{k}{2}(x_1^2 + x_2^2).$$

i) Derive the equations of motion.

ii) Use the definition for the Poisson bracket to show that the two quantities

$$L(x_1, x_2, p_1, p_2) = x_1 p_2 - x_2 p_1$$
$$K(x_1, x_2, p_1, p_2) = \frac{1}{2m}(p_1^2 - p_2^2) + \frac{k}{2}(x_1^2 - x_2^2)$$

are conserved in time, i.e. $\dot{L} = \dot{K} = 0$.

iii) Employ the Jacobi-Poisson theorem to construct a new conserved quantity from L and K.

iv) Is the conserved quantity constructed in *iii)* independent of the previous ones?