# CW 2 - 2009 (Solutions)

1) The fixed points are found from

$$F(x,\beta) = x^3 + \alpha x^2 + \beta x = 0$$

$\Rightarrow$ three curves $\quad x_1 = 0 \qquad x_{2/3} = \frac{1}{2}\left(-\alpha \pm \sqrt{\alpha^2 - 4\beta}\right)$

$\Rightarrow \quad \frac{\partial F}{\partial x} = 3x^2 + 2\alpha x + \beta \qquad \frac{\partial F}{\partial \beta} = x$ (1)

- pitchfork bifurcation ($\alpha = 0$)

$$\left.\frac{\partial F}{\partial \beta}\right|_{(x_0,\beta_0)} = 0 = x \Rightarrow \underline{x_0 = 0} \qquad \left.\frac{\partial F}{\partial x}\right|_{(x_0,\beta_0)} = 3x_0^2 + 2\alpha x_0 + \beta_0 = 0 \Rightarrow \underline{\beta_0 = 0}$$

$$F(x,\beta) = 0 = x(x^2+\beta) \Rightarrow \beta = -x^2 \Rightarrow \frac{\partial \beta}{\partial x} = -2x$$

$\Rightarrow$ changes sign at $x_0 = 0$

$\Rightarrow$ tangent are different at diff. curve

$\Rightarrow$ pitchfork bifurcation at $(x_0, \beta_0) = (0,0)$ (3)

- transcritical bifurcation

for $\alpha \neq 0$: $\quad F(x,\beta) = x(x^2 + \alpha x + \beta) = 0$

$\Rightarrow \quad \beta = -x\alpha - x^2$

$\Rightarrow \quad \frac{\partial \beta}{\partial x} = -\alpha - 2x \Rightarrow$ does not change sign at $x_0 = 0$

$\therefore$ two branches pass through $(0,0)$, with distinct tangent

$\Rightarrow$ $(0,0) = (x_0, \beta_0)$ is a transcritical bifurcation (3)

- turning point

$$\frac{\partial x_{2/3}}{\partial \beta} = \frac{\mp \beta}{\sqrt{\alpha^2 - 4\beta}} \Rightarrow \frac{\partial x}{\partial \beta} \text{ changes sign for } \beta_0 = \frac{1}{4}\alpha^2$$

$$\Rightarrow x_2(\beta_0) = x_3(\beta_0) = -\frac{\alpha}{2} = x_0$$

$$\Rightarrow \left.\frac{\partial F}{\partial \beta}\right|_{(\beta_0, x_0)} = x_0 = -\frac{\alpha}{2}$$ (3)

$\Rightarrow$ there is a turning point at $(x_0, \beta_0) = \left(-\frac{\alpha}{2}, \frac{\alpha^2}{4}\right)$ $\frac{10}{10}$

2)

$$\ddot{x} + \lambda(x^2-1)\dot{x} + x = 0$$

$$\left.\begin{array}{l} x_1 = x \\ x_2 = \dot{x} \end{array}\right\} \Rightarrow \quad \begin{array}{l} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\lambda(x_1^2-1)x_2 - x_1 \end{array}$$

fixed point at $(x_1, x_2) = (0, 0)$

linearisation:

$$J = \begin{pmatrix} 0 & 1 \\ -1 & \lambda \end{pmatrix} \quad \Rightarrow \text{ eigenvalues } \quad e_\pm = \frac{1}{2}\left(\lambda \pm \sqrt{\lambda^2-4}\right)$$

$$\Rightarrow \quad \left.\begin{array}{lll} 0 < \lambda < 2 & \Rightarrow & \text{unstable focus} \\ \lambda = 0 & \Rightarrow & \text{centre} \\ -2 < \lambda < 0 & \Rightarrow & \text{stable focus} \end{array}\right\} \Rightarrow \text{ Hopf bifurcation}$$

The theorem from the lecture can not be applied:

Theorem: (details see lecture notes)

i) for $\bar{\lambda} = \lambda = 0$ the linearised system has purely imaginary eigenvalues

ii) $\left.\frac{d}{d\lambda} \operatorname{Re}(e_\pm(\lambda))\right|_{\lambda=\bar{\lambda}=0} = \frac{1}{2} > 0$

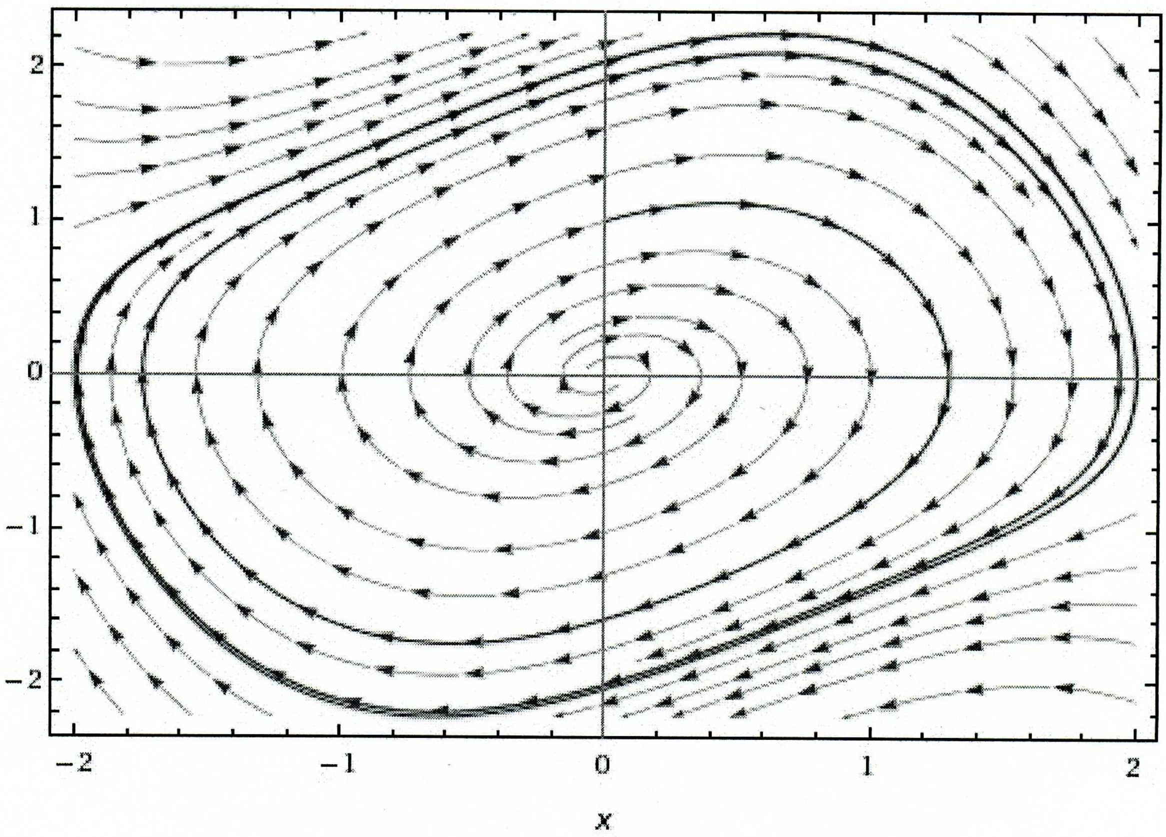
iii) the origin should be asymptotically stable

This point is difficult to show as the index becomes $I = 0$ for $\bar{\lambda} = 0 = \lambda$.

$\Rightarrow$ The qualitative analysis of the linearised system is enough in this case.

10

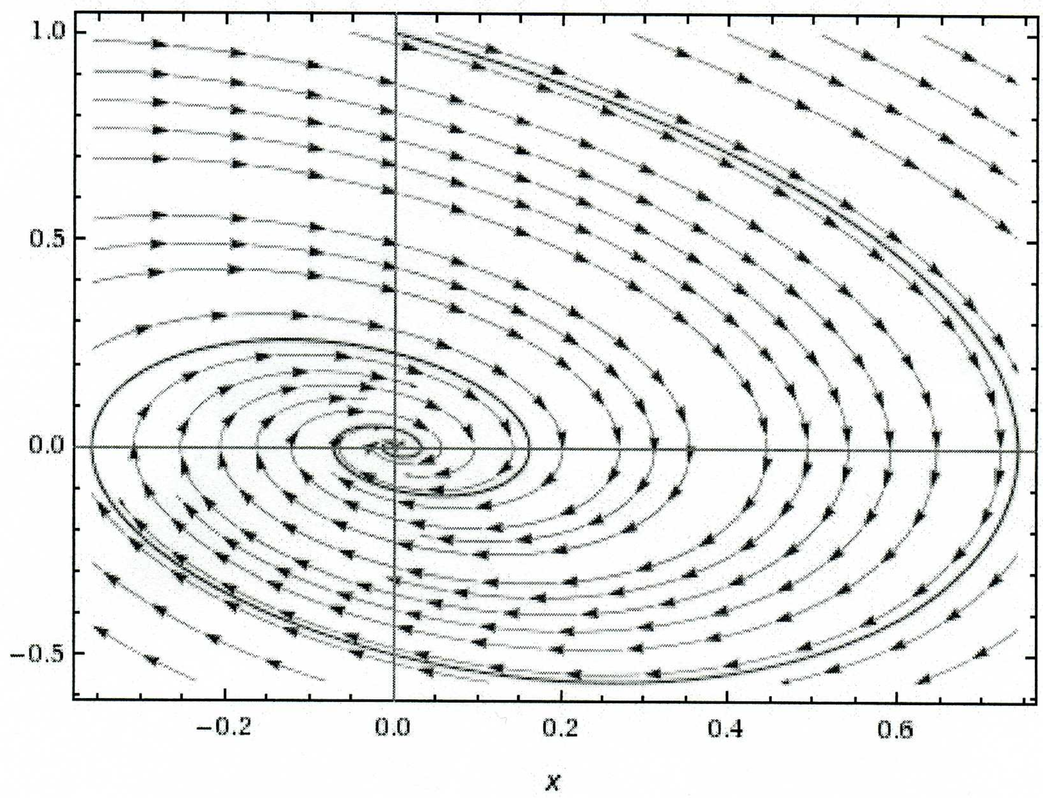
Phase portrait with limit cycle:




$\lambda = 0.5$

(the critical point origin is an unstable focus)

Phase portrait with limit cycle:




$\lambda = -0.5$

(the critical point origin is a stable focus)

5

15/15

3) i)

$$H(x_1, x_2) = \frac{1}{2} x_2^2 + \underbrace{k\, e^{-x_1} \sin x_1}_{V(x_1)}$$

$$\Rightarrow \quad \dot{x}_1 = \frac{\partial H}{\partial x_2} = x_2 \qquad \dot{x}_2 = -\frac{\partial H}{\partial x_1} = -\frac{\partial V}{\partial x_1}$$

Compute: $V(x_1) = k e^{-x_1} \sin x_1$

$V'(x_1) = k e^{-x_1} (\cos x_1 - \sin x_1)$

$V''(x_1) = -2k\, e^{-x_1} \cos x_1$

stationary points from $V'(x_1) = 0 \Rightarrow \underline{x_1^{(n)} = \frac{\pi}{4} + n\pi}$ $\quad n \in \mathbb{Z}$

$$\Rightarrow V''(x_1) = -2k\, e^{-(\frac{\pi}{4} + n\pi)} \cos(\frac{\pi}{4} + n\pi) = (-1)^{n+1} \sqrt{2}\, k\, e^{-(\frac{\pi}{4} + n\pi)}$$

$\Rightarrow$ for $k \in \mathbb{R}^+$, $n$ even or $k \in \mathbb{R}^-$, $n$ odd $\Rightarrow V''(x_1^{(n)}) < 0$

$\Rightarrow$ maximum at $x_1^{(n)}$ $\Rightarrow$ saddle point at $(x_1^{(n)}, 0)$

$\Rightarrow$ for $k \in \mathbb{R}^+$, $n$ odd or $k \in \mathbb{R}^-$, $n$ even $\Rightarrow V''(x_1^{(n)}) > 0$

$\Rightarrow$ minimum at $x_1^{(n)}$ $\Rightarrow$ centre at $(x_1^{(n)}, 0)$ 5

ii) The period is computed from

$$T = 2 \int_{\alpha}^{\beta} \frac{dx}{\sqrt{2(E - V(x))}}$$

Compute $E$:

$$H(2^{\frac{2}{4}}, 2) = \frac{1}{2} 2^2 + \frac{1}{4} 2^3 = \underline{4 = E}$$

Compute turning points:

$$E = 4 = H(x_t, 0) = \frac{1}{4} x_t^4 = 4 \quad \Rightarrow \quad \underline{x_t^{(1/2)} = \pm 2}$$

$$\Rightarrow T = 2 \int_{-2}^{2} \frac{dx}{\sqrt{2(4 - x^4/4)}} = 4 \int_0^2 \frac{dx}{\sqrt{2(4 - x^4/2)}} = 4 \int_0^1 \frac{2\, dx}{\sqrt{8(1 - x^4)}}$$

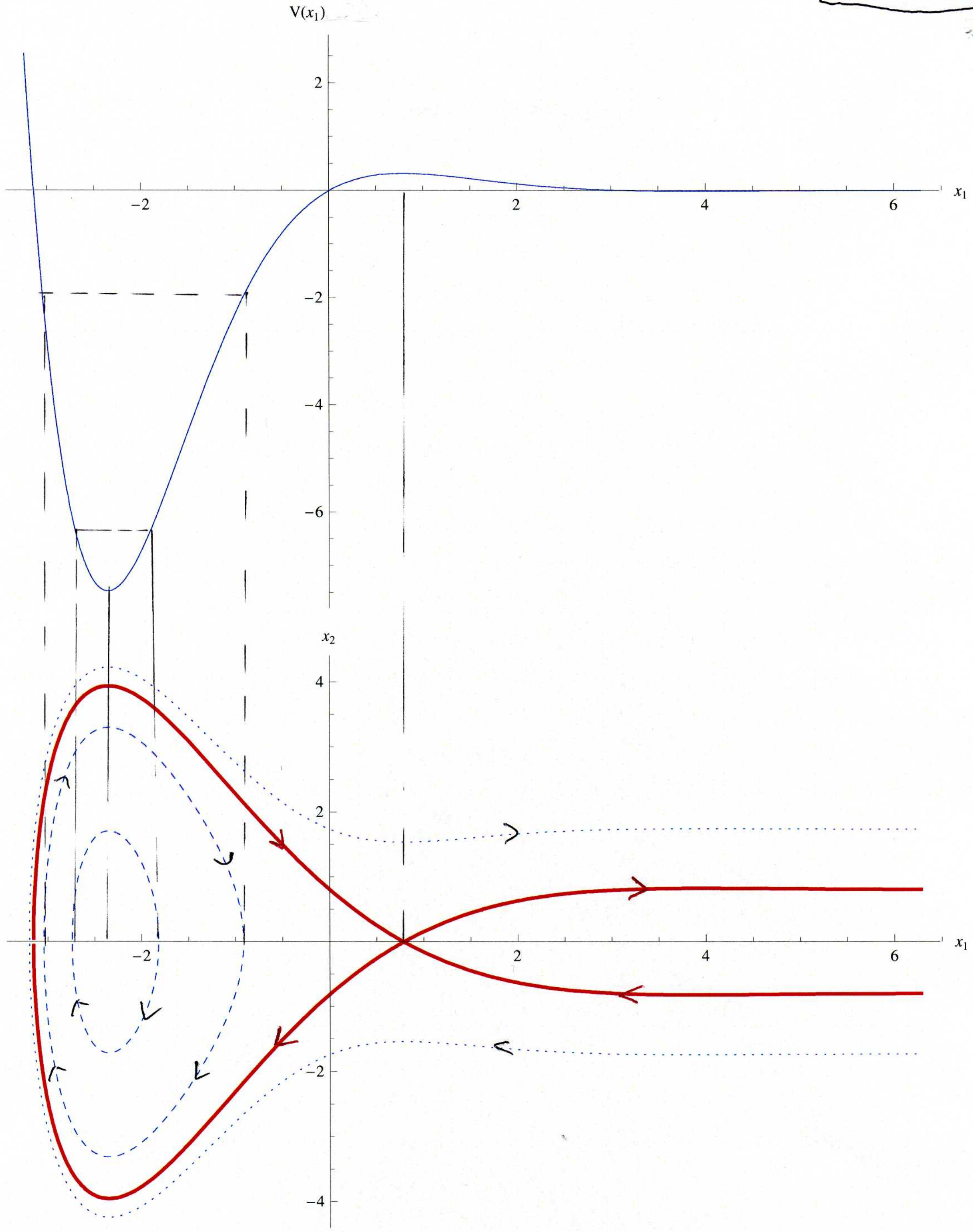
$$= \sqrt{8} \int_0^1 \frac{dx}{\sqrt{1 - x^4}}$$

$$\underline{T = \sqrt{8\pi}\, \frac{\Gamma(\frac{5}{4})}{\Gamma(\frac{3}{4})}}$$

5

3 ii)

$\boxed{\kappa = 1 > 0}$

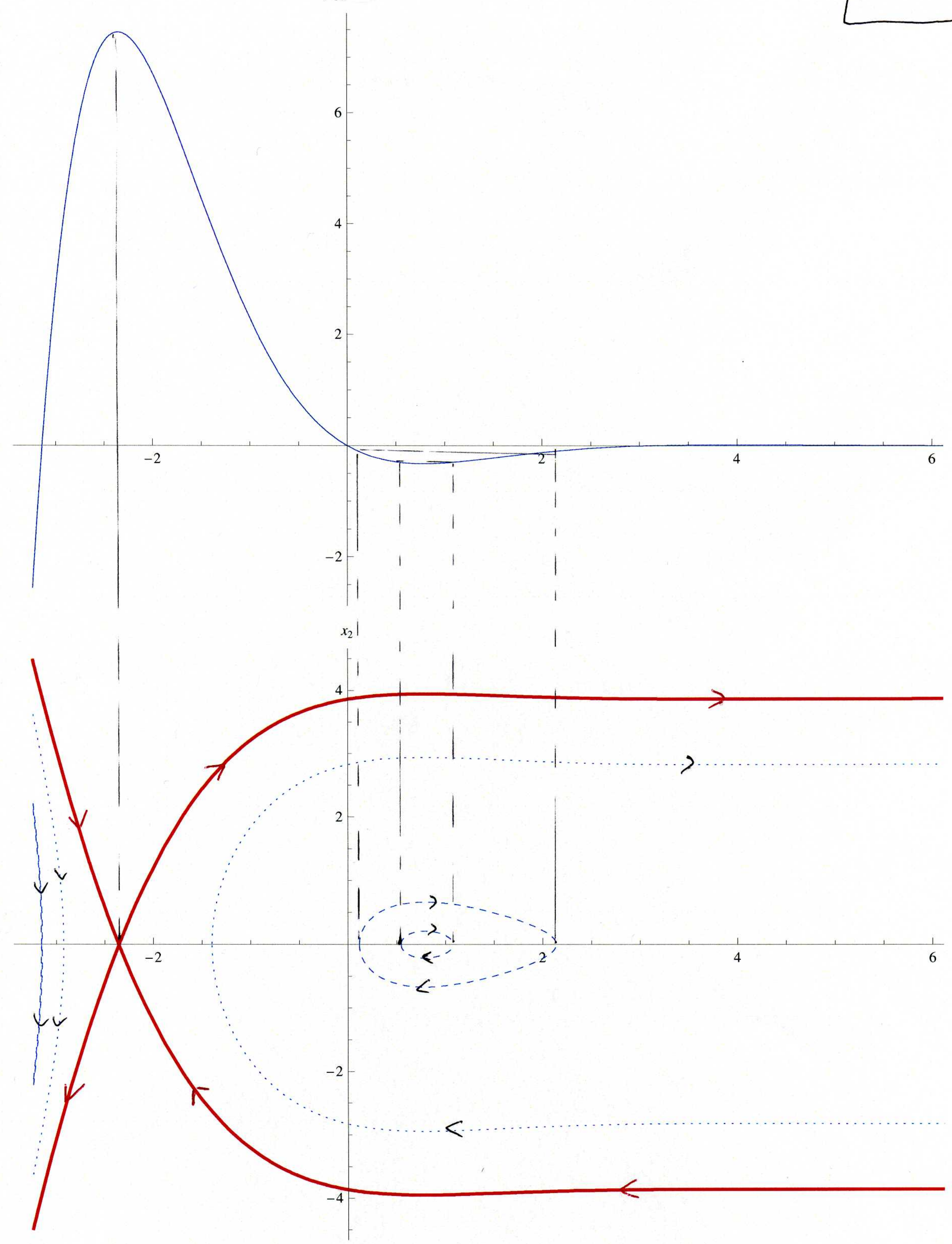



time direction from $\dot{x}_1 = x_2$, i.e. $\rightarrow$ in the upper half plane

$\leftarrow$ in the lower half plane

3 ii)

$\boxed{K = -1 < 0}$




time direction from $\dot{x}_1 = x_2$ i.e. $\rightarrow$ in the upper half plane

$\leftarrow$ in the lower half plane

4)

$$H(x_1, x_2, p_1, p_2) = \frac{1}{2m}\left(p_1^2 + p_2^2\right) + \frac{k}{2}\left(x_1^2 + x_2^2\right)$$

i) $\dfrac{\partial H}{\partial x_1} = k x_1 = -\dot{p}_1 \qquad \dfrac{\partial H}{\partial x_2} = k x_2 = -\dot{p}_2$

$\dfrac{\partial H}{\partial p_1} = \dfrac{1}{m} p_1 = \dot{x}_1 \qquad \dfrac{\partial H}{\partial p_2} = \dfrac{1}{m} p_2 = \dot{x}_2$ (1)

ii) Using the definition of the Poisson bracket

$$\{f, g\} = \sum_{i=1}^{n/2}\left(\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial x_i}\right)$$

- for $L = x_1 p_2 - x_2 p_1$

$$\Rightarrow \{L, H\} = \frac{\partial L}{\partial x_1}\frac{\partial H}{\partial p_1} - \frac{\partial L}{\partial p_1}\frac{\partial H}{\partial x_1} + \frac{\partial L}{\partial x_2}\frac{\partial H}{\partial p_2} - \frac{\partial L}{\partial p_2}\frac{\partial H}{\partial x_2}$$

$$= p_2 \frac{1}{m} p_1 + x_2 k x_1 - p_1 \frac{1}{m} p_2 - x_1 k x_2 = 0$$

$\Rightarrow L$ is conserved in time

- for $K = \frac{1}{2m}\left(p_1^2 - p_2^2\right) + \frac{k}{2}\left(x_1^2 - x_2^2\right)$

$$\Rightarrow \{K, H\} = \frac{\partial K}{\partial x_1}\frac{\partial H}{\partial p_1} - \frac{\partial K}{\partial p_1}\frac{\partial H}{\partial x_1} + \frac{\partial K}{\partial x_2}\frac{\partial H}{\partial p_2} - \frac{\partial K}{\partial p_2}\frac{\partial H}{\partial x_2}$$

$$= k x_1 \frac{1}{m} p_1 - \frac{1}{m} p_1 k x_1 - k x_2 \frac{1}{m} p_2 + \frac{1}{m} p_2 k x_2 = 0$$

$\Rightarrow K$ is conserved in time (4)

iii)

$$\{L, K\} = \frac{\partial L}{\partial x_1}\frac{\partial K}{\partial p_1} - \frac{\partial L}{\partial p_1}\frac{\partial K}{\partial x_1} + \frac{\partial L}{\partial x_2}\frac{\partial K}{\partial p_2} - \frac{\partial L}{\partial p_2}\frac{\partial K}{\partial x_2}$$

$$= p_2 \frac{1}{m} p_1 + x_2 k x_1 + p_1 \frac{1}{m} p_2 + x_1 k x_2$$

$$= \frac{2}{m} p_1 p_2 + 2 k x_1 x_2 =: M$$

$$\Rightarrow \{M, H\} = \frac{\partial M}{\partial x_1}\frac{\partial H}{\partial p_1} - \frac{\partial M}{\partial p_1}\frac{\partial H}{\partial x_1} + \frac{\partial M}{\partial x_2}\frac{\partial H}{\partial p_2} - \frac{\partial M}{\partial p_2}\frac{\partial H}{\partial x_2}$$ (2)

$$= 2 k x_2 \frac{1}{m} p_1 - \frac{2}{m} p_2 k x_1 + 2 k x_1 \frac{1}{m} p_2 - \frac{2}{m} p_1 k x_2 = 0$$

$\Rightarrow \{L, K\} =: M$ is conserved in time

iv)

$$H^2 = \left[\frac{1}{2m}\left(p_1^2 + p_2^2\right) + \frac{k}{2}\left(x_1^2 + x_2^2\right)\right]^2$$

$$= \frac{p_1^4}{4m^2} + \frac{p_2^4}{4m^2} + \frac{p_1^2 p_2^2}{2m^2} + \frac{k}{2m}\left(p_1^2 x_1^2 + p_2^2 x_1^2\right) + \frac{1}{4} k^2 x_1^4 + \frac{1}{4} k^2 x_2^4$$

$$+ \frac{1}{2} k^2 x_1^2 x_2^2 + k \frac{p_1^2 x_2^2}{2m} + k \frac{p_2^2 x_2^2}{2m}$$

$$K^2 = \frac{p_1^4}{4m^2} + \frac{p_2^4}{4m^2} - \frac{p_1^2 p_2^2}{2m^2} + \frac{k}{2m}\left(p_1^2 x_1^2 - p_2^2 x_1^2\right) + \frac{1}{4} k^2 x_1^4 + \frac{1}{4} k^2 x_2^4$$

$$- \frac{1}{2} k^2 x_1^2 x_2^2 - k \frac{p_1^2 x_2^2}{2m} + k \frac{p_2^2 x_2^2}{2m}$$

$$\frac{1}{4} M^2 = \frac{p_1^2 p_2^2}{m^2} + \frac{2k}{m} p_1 p_2 x_1 x_2 + k^2 x_1^2 x_2^2$$

3

$$\frac{k}{m} L^2 = \frac{k}{m}\left(p_2^2 x_1^2 + p_1^2 x_2^2 - 2 p_1 p_2 x_1 x_2\right)$$

$$\Rightarrow \quad H^2 = K^2 + \frac{1}{4} M^2 + \frac{k}{m} L^2$$

10/10