Dynamical Systems II

Coursework 2

Hand in the complete solutions to all three questions in the SEMS general office (C108).

DEADLINE: Thursday 09/12/2010 at 16:00

1. (i) Describe what is meant by bifurcation theory, a bifurcation and a bifurcation diagram. Consider the one dimensional system

$$\dot{x} = x^3 + \gamma x^2 - \lambda x \quad \text{with } \gamma, \lambda \in \mathbb{R}.$$

Find the fixed points for the system. For $\gamma = 0$ find the pitchfork deformation point for the above system. Show that when $\gamma \neq 0$ this point transforms into a transcritical bifurcation point. Find the turning point of the system as a function of γ.

(ii) Consider the dynamical system

$$\begin{aligned} \dot{x}_1 &= 9x_2 + 3x_1^2 \\ \dot{x}_2 &= \lambda x_2 - 2x_1^2 x_2 - 9x_1 - 2x_1^3 + \alpha x_1^2 \end{aligned}$$

with $\lambda \in \mathbb{R}$ being a bifurcation parameter and constant $\alpha \in \mathbb{R}$. For which values of α is the origin an asymptotically stable fixed point?

Hint: You may use the stability index

$$\begin{aligned} I =\ & \omega \left(Y^1_{111} + Y^1_{122} + Y^2_{112} + Y^2_{222} \right) + Y^1_{11} Y^2_{11} - Y^1_{11} Y^1_{12} + Y^2_{11} Y^2_{12} \\ & + Y^2_{22} Y^2_{12} - Y^1_{22} Y^1_{12} - Y^1_{22} Y^2_{22} \end{aligned}$$

where

$$Y^i_{jk} = \left. \frac{\partial^2 Y_i}{\partial y_j \partial y_k} \right|_{(0,0)} \qquad \text{and} \qquad Y^i_{jkl} = \left. \frac{\partial^3 Y_i}{\partial y_j \partial y_k \partial y_l} \right|_{(0,0)}.$$

(iii) State the Hopf bifurcation theorem and use it to decide whether the system in (ii) possesses a Hopf bifurcation point for $\lambda = 0$ when $\alpha = 2$ and $\alpha = 4$.

2. (i) What is meant by a Hamiltonian system in two dimensions, equations of motion and an autonomous system?

(*ii*) Provide a criterium that can be used to decide whether a system is Hamiltonian or not. Employ the criterium to determine the value of μ such that the system

$$\begin{aligned}\dot{x}_1 &= 3x_1^2x_2^2 + 2x_1 + 5x_2 \\ \dot{x}_2 &= -\mu x_1^2x_2^3 - 2x_2 + \sin(x_1^5)\end{aligned}$$

becomes a Hamiltonian system.

(*iii*) Provide a definition for a potential system. Derive the Hamiltonian function for the dynamical system

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -2x_1 + \frac{20x_1}{1+x_1^2}\end{aligned}$$

and confirm that it is a potential system. Fix the potential in such way that its value at zero is zero.

(*iv*) Compute all fixed points of the system in (*iii*) and determine their nature by making use of the fact that the system is a potential system.

(*v*) Sketch the potential for the system in (*iii*) and compute the equation for the separatrices by making use of the fact that the Hamiltonian is conserved along a trajectory. Draw the phase portrait for the system. Include the separatrices and some representative trajectories. Include the direction of time on the trajectories and provide a reasoning for your choices. Explain in which areas of your diagram the motion is bounded.

3. Consider the following difference equation

$$x_{n+1} = F(x_n) = 8\lambda x_n - 4\lambda x_n^2 \qquad \text{for } \lambda \in \mathbb{R}^+.$$

λ is taken to be the bifurcation parameter.

(*i*) Depending on the values of λ, determine the nature of the fixed points and their stability.

(*ii*) State the condition which determines the existence of a 2-cycle. Show that 2-cycles for the above system are governed by the solutions of the equation

$$1 + 8\lambda - 4x\lambda - 32x\lambda^2 + 16x^2\lambda^2 = 0 \ .$$

Compute the solution of this equation and use it to argue that the existence of a 2-cycle requires $\lambda \geq 3/8$.

(*iii*) Determine the domain of stability for the 2-cycle.