Dynamical Systems II

Solutions and marking scheme for coursework 1

INSTRUCTIONS: Each question carries 20 marks.

1. (*i*) Defining [2]

$$x_1 = x \qquad \text{and} \qquad x_2 = \dot{x},$$

we obtain

$$\dot{x}_1 = \dot{x} = x_2,$$
$$\dot{x}_2 = \ddot{x} = -x_2 - \mu x_1^3 - \nu x_2^5.$$

(*ii*) The fixed point $\vec{x}_f = (0, 0)$ results from [1]

$$x_2 = 0,$$
$$-x_2 - \mu x_1^3 - \nu x_2^5 = 0.$$

Linearization theorem: *Consider a nonlinear system which possesses a simple linearization at some fixed point. Then in a neighbourhood of the fixed point the phase portraits of the linear system and its linearization are qualitatively equivalent, if the eigenvalues of the Jacobian matrix have a nonzero real part, i.e. the linearized system is not a centre.* [2]

We compute the Jacobian for the above system at the fixed point:

$$A(\vec{x}_f) = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \Rightarrow \det A(\vec{x}_f) = 0 \Rightarrow \text{non-simple linearization}$$

The linearization theorem can not be applied since the system is non-simple. [3]

(*iii*) **Lyapunov stability theorem:** *Consider the system $\dot{\vec{x}} = \vec{F}(\vec{x})$ with a fixed point at the origin. If there exists a real valued function $V(\vec{x})$ in a neighbourhood $N(\vec{x} = 0)$ such that:*

i) *the partial derivatives $\partial V/\partial x_1$, $\partial V/\partial x_2$ exist and are continuous,*

ii) *the function $V(\vec{x})$ is positive definite,*

iii) *dV/dt is negative semi-definite (definite),* *then the origin is a stable (asymptotically stable) fixed point.*

Verify conditions: 5

i) the partial derivatives $\partial V/\partial x_1 = 4\alpha x_1^3$, $\partial V/\partial x_2 = 4x_2$ exist and are continuous,

ii) the function $V(\vec{x})$ is positive definite, i.e. $V(\vec{0}) = 0$ and $V(\vec{x}) > 0 \forall \vec{x} \neq \vec{0}$,

iii) dV/dt should be negative semi-definite for $V(\vec{x})$ to be a weak Lyapunov function:

$$\begin{aligned}\frac{dV}{dt} &= \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2 \\ &= 4\alpha x_1^3 x_2 + 4x_2(-x_2 - \mu x_1^3 - \nu x_2^5) \\ &= (4\alpha - 4\mu)x_1^3 x_2 - 4x_2^2 - 4\nu x_2^6\end{aligned}$$

$\Rightarrow dV/dt$ should be negative semi-definite for $\alpha = \mu$ and $\nu \geq 0$.

(iv) The partial derivatives $\partial V_1/\partial x_1$ and $\partial V_1/\partial x_2$ exist and are continuous. $V_1(\vec{x})$ is positive definite. But 3

$$\begin{aligned}\frac{dV}{dt} &= \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2 \\ &= 2x_1 x_2 + 2x_2(-x_2 - x_1^3)\end{aligned}$$

does not lead to a negative semi-definite function. Therefore V_1 is not a Lyapunov function.

$V_2(\vec{x})$ is not positive definite. Therefore V_2 is not a Lyapunov function.

(v) **Corollary:** *Let $V[\vec{x}(t)]$ be a weak Lyapunov function for the system $\dot{\vec{x}} = \vec{F}(\vec{x})$ in a neighbourhood of the isolated fixed point $\vec{x}_f = (0,0)$. Then if $\dot{V} \neq 0$ on any trajectory, except for the fixed point, the origin is asymptotically stable.* 4

- We have $dV/dt = 0$ for $\vec{x} = (x_1, 0)$.
- On this line we have $\dot{x}_1 = 0$ and $\dot{x}_2 = -\mu x_1^3$, which means the line $\vec{x} = (x_1, 0)$ is not a trajectory.
- Therefore $\vec{x} = (0,0)$ is asymptotically stable.

$\sum = 20$

2. We have

$$\begin{aligned}\dot{x}_1 &= x_2 + x_1(x_1^2 + x_2^2 - 5)(1 - x_1^4 - x_2^4 - 2x_1^2 x_2^2) \\ \dot{x}_2 &= -x_1 + x_2(x_1^2 + x_2^2 - 5)(1 - x_1^4 - x_2^4 - 2x_1^2 x_2^2)\end{aligned}$$

(i) With $x_1 = r\cos\vartheta$ and $x_2 = r\sin\vartheta$ we obtain 4

$$\dot{x}_1 = \dot{r}\cos\vartheta - r\sin\vartheta\dot{\vartheta} = r\sin\vartheta + r\cos\vartheta(r^2 - 5)(1 - r^4) \quad (1)$$

$$\dot{x}_2 = \dot{r}\sin\vartheta + r\cos\vartheta\dot{\vartheta} = -r\cos\vartheta + r\sin\vartheta(r^2 - 5)(1 - r^4) \quad (2)$$

where $x_1^2 + x_2^2 = r^2$. Computing (1) $\times \cos\vartheta$ + (2) $\times \sin\vartheta$ gives

$$\dot{r} = r(r^2 - 5)(1 - r^4). \quad (3)$$

Next we compute (1) $\times \sin\vartheta$ - (2) $\times \cos\vartheta$:

$$-r\dot{\vartheta} = r$$

Dividing by r gives

$$\dot{\vartheta} = -1. \tag{4}$$

Since $\dot{\vartheta} \neq 0$ the origin is the only fixed point. 1

(*ii*) **Poincaré-Bendixson theorem:** *Let φ_t be a flow for the system $\dot{\vec{x}} = \vec{F}(\vec{x})$ and let $\mathcal{D}$ be a closed, bounded and connected set $\mathcal{D} \in \mathbb{R}^2$, such that $\varphi_t(\mathcal{D}) \subset \mathcal{D}$ for all time. Furthermore $\mathcal{D}$ does not contain any fixed point. Then there exists at least one limit cycle in $\mathcal{D}$.* 2

For $r = 2$ we compute 3

$$\dot{r}(2) = 2(4-5)(1-16) = 30 > 0,$$

and for $r = 3$ we compute

$$\dot{r}(3) = 2(9-5)(1-81) = -960 < 0.$$

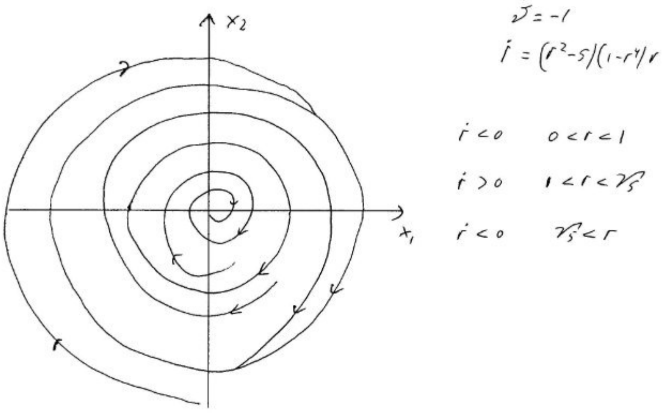
$\Rightarrow$ trajectories which enter the region

$$\mathcal{D} = \{(r, \vartheta) : 2 \leq r \leq 3\}$$

can never leave it.

$\Rightarrow$ Since there is no fixed point in $\mathcal{D}$, see (*i*), we can employ the Poincaré-Bendixson theorem to deduce that there is at least one limit cycle in $\mathcal{D}$.

(*iii*) Equations (3) and (4) give the diagram: 2



(*iv*) **Def.:** *The* *<u>ω-limit set</u>* *(or positive limit set) $L_\omega(\vec{x})$ of a point $\vec{x}$ contains those points which are approached by the trajectory through $\vec{x}$ as $t \to \infty$, that is* 1

$$L_\omega(\vec{x}) = \{\vec{y} \in \mathbb{R}^n : \exists \text{ a sequence of times } t_n \text{ with } t_n \to \infty, \text{ such that } \lim_{n\to\infty} \varphi_{t_n}(\vec{x}) = \vec{y}\Big\}$$

Def.: *The α-limit set (or negative limit set) $L_\alpha(\vec{x})$ of a point $\vec{x}$ contains those points which are approached by the trajectory through $\vec{x}$ as $t \to -\infty$, that is*

$$L_\alpha(\vec{x}) = \Big\{\vec{y} \in \mathbb{R}^n : \exists \text{ a sequence of times } t_n \text{ with } t_n \to -\infty, \text{ such that } \lim_{n\to\infty} \varphi_{t_n}(\vec{x}) = \vec{y}\Big\}$$

Accordingly we compute for (3) and (4) 3

$$L_\alpha(\vec{x}) = \begin{cases} 0 & \text{for } r = 0 \\ \mathcal{C}_1 & \text{for } 0 < r < \sqrt{5} \\ \mathcal{C}_{\sqrt{5}} & \text{for } r = \sqrt{5} \\ \varnothing & \sqrt{5} < r \end{cases} \qquad L_\omega(\vec{x}) = \begin{cases} 0 & \text{for } 0 \leq r < 1 \\ \mathcal{C}_1 & \text{for } r = 1 \\ \mathcal{C}_{\sqrt{5}} & \text{for } 1 < r \end{cases}$$

Def.: *A closed orbit ϕ is a limit cycle if ϕ is a subset of an α or ω-limit set for some point $\vec{x} \notin \phi$.* 2

We have $\dot{r} = 0$ for $r = 0, \sqrt{5}, 1$, which means we have a limit cycle with radius $r = 1 : \mathcal{C}_1$ and one with radius $r = \sqrt{5} : \mathcal{C}_{\sqrt{5}}$.

Def.: *A limit cycle ϕ is a called a stable (unstable) limit cycle, if $\phi = L_\omega(\vec{x})$ ($\phi = L_\alpha(\vec{x})$) for all $\vec{x}$ in some neighbourhood of the limit cycle.*

Def.: *A limit cycle ϕ is a called a semi-stable limit cycle, if it is a stable limit cycle for points on one side and an unstable limit cycle for point on the other side.*

Therefore $\mathcal{C}_1$ is unstable and $\mathcal{C}_{\sqrt{5}}$ is stable. The limit cycle $\mathcal{C}_{\sqrt{5}}$ is the one identified in (*ii*) since $\mathcal{C}_{\sqrt{5}} \subset \mathcal{D}$.

(v) **Bendixson's criterium:** *Let $\mathcal{D}$ be a simply connected region of the phase plane in which the function $\vec{F}(\vec{x})$ of the system $\dot{\vec{x}} = \vec{F}(\vec{x})$ has the property that its divergence is of constant sign, i.e.* 2

$$\operatorname{div} \vec{F} = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} < 0 \qquad \text{or} \qquad \operatorname{div} \vec{F} = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} > 0.$$

Then the system posesses no closed orbit contained entirely in $\mathcal{D}$.

$\hat{\mathcal{D}}$ is not a simply connected region and therefore we can not apply Bendixson's criterium to decide whether it contains limit cycles or not. $\sum = 20$