Dynamical Systems II

Solutions and marking scheme for coursework 2

INSTRUCTIONS: Each question carries 20 marks.

1) *i)* We have the following **corollary:** *Suppose that for the system $\dot{\vec{x}} = \vec{X}(\vec{x})$ we have transformed the linearized system $\dot{\vec{x}} = A\vec{x}$ with the help of $\vec{x} = U\vec{y}$ into the Jordan normal form*

$$\dot{\vec{y}} = U^{-1}AU\vec{y} = J\vec{y} = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}, \qquad (1)$$

where $\vec{x} = U\vec{y}$, $\dot{\vec{y}} = \vec{Y}(\vec{y})$. Then if the stability index

$$I = \omega\left(Y^1_{111} + Y^1_{122} + Y^2_{112} + Y^2_{222}\right) + Y^1_{11}(Y^2_{11} - Y^1_{12}) + Y^2_{22}(Y^2_{12} - Y^1_{22}) + Y^2_{11}Y^2_{12} - Y^1_{22}Y^1_{12}$$

computed from (1) and $\dot{\vec{y}} = \vec{Y}(\vec{y})$ is negative, the origin is asymptotically stable.

We compute the Jacobian for the system

$$\dot{x}_1 = 7x_2 \qquad \dot{x}_2 = -(x_1^2 - \lambda)x_2 - 7x_1 - 2x_1^3$$

to

$$A = \begin{pmatrix} 0 & 7 \\ -7 & 0 \end{pmatrix}.$$

Note this is already in Jordan normal form, such that $A = J$ and $\vec{X} = \vec{Y}$. Therefore $\omega = 7$. The only nonvanishing term in I is $Y^2_{112} = -2$. This means

$$I = \omega Y^2_{112} = -14.$$

As I is negative it follows that the origin is asymptotically stable. 5

ii) **Hopf bifurcation theorem:** *Let (0,0,λ) with $\lambda \in \mathbb{R}$ be a fixed point of the system*

$$\dot{x}_1 = F(x_1, x_2, \lambda)$$
$$\dot{x}_2 = G(x_1, x_2, \lambda).$$

If

i) The eigenvalues $e_1(\lambda)$ and $e_2(\lambda)$ of the linearized system are purely imaginary for some value $\lambda = \tilde{\lambda}$, i.e. $e_1(\lambda) \in i\mathbb{R}$ and $e_2(\lambda) \in i\mathbb{R}$.

ii) The real part of the eigenvalues $\mathrm{Re}(e_{1/2}(\lambda))$ satisfies

$$\frac{d}{d\lambda} \mathrm{Re}(e_{1/2}(\lambda)) \Big|_{\lambda=\tilde{\lambda}} > 0.$$

iii) The origin is asymptotically stable for $\lambda = \tilde{\lambda}$.

then

a) $\lambda = \tilde{\lambda}$ is a bifurcation point of the system.

b) For $\lambda \in (\lambda_1, \tilde{\lambda})$ with some $\lambda_1 < \tilde{\lambda}$ the origin is a stable focus.

c) For $\lambda \in (\tilde{\lambda}, \lambda_2)$ with some $\lambda_2 > \tilde{\lambda}$ the origin is an unstable focus surrounded by a stable limit cycle whose size increases with λ. State the Hopf bifurcation theorem and use it to prove that the system possesses a Hopf bifurcation for $\lambda = 0$. 3

The Jacobian matrix for $\lambda \neq 0$ is

$$A = \begin{pmatrix} 0 & 7 \\ -7 & \lambda \end{pmatrix},$$

with eigenvalues $e_\pm = \lambda/2 \pm \sqrt{\lambda^2 - 196}$.

i) for $\lambda = 0$ the eigenvales are purely imaginary $e_\pm = \pm i7$.

ii) we compute

$$\frac{d}{d\lambda} \mathrm{Re}(e_{1/2}(\lambda)) \Big|_{\lambda=\tilde{\lambda}=0} = \frac{1}{2} > 0.$$

iii) from part *i)* of the question we know that the origin is asymptotically stable.

Therefore the Hopf bifurcation theorem applies. 3

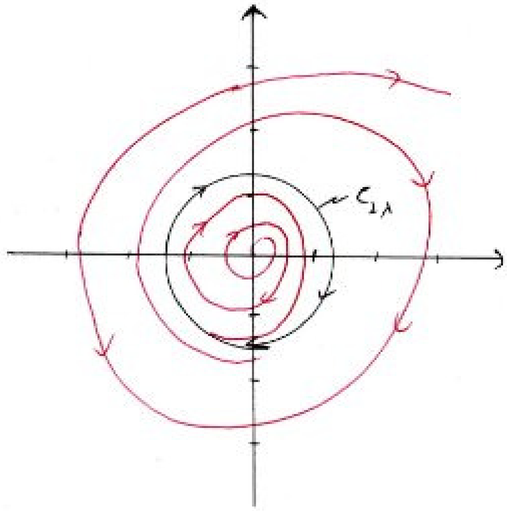
iii) We have

$$\dot{r} = 0 \quad \text{for } r = 0, 2\lambda \qquad \text{and} \qquad \dot{r} > 0 \quad \text{for } r \neq 0, 2\lambda$$

Therefore

$$L_\alpha(\vec{x}) = \begin{cases} 0 & \text{for } 0 \leq r < 2\lambda \\ C_{2\lambda} & \text{for } r \neq 2\lambda \end{cases} \qquad L_\omega(\vec{x}) = \begin{cases} 0 & \text{for } r = 0 \\ C_{2\lambda} & \text{for } 0 < r \leq 2\lambda \\ \varnothing & \text{for } r > 2\lambda \end{cases}$$

Phase portrait:



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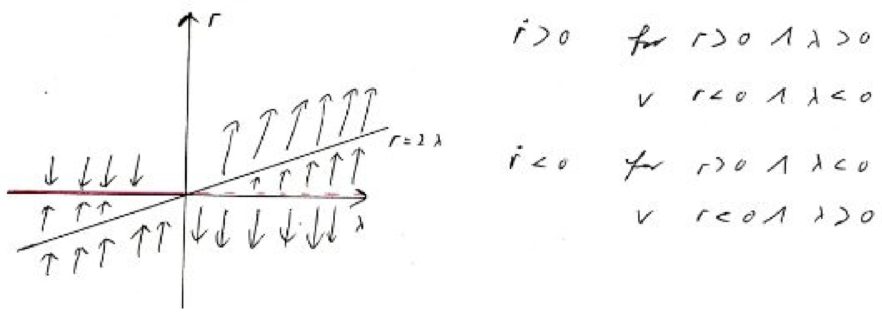
The fixed points are at $r = 0$ and $r = 2\lambda$. With $F(r, \lambda)$ follows

$$\frac{\partial F}{\partial r} = 4\lambda^3 - 8r\lambda^2 + 3r^2\lambda = 0 \quad \Rightarrow \lambda = 0$$
$$\frac{\partial F}{\partial \lambda} = r^3 - 8r^2\lambda + 12r\lambda^2 = 0 \quad \Rightarrow r = 0$$

which means there is a transcritical bifurcation point at $(0, 0)$, which follows from the definition: *Let* (x_0, λ_0) *be a fixed point for the system* $\dot{x} = F(x, \lambda)$. *If* $\partial F/\partial\lambda|_{(x_0,\lambda_0)} = 0$ *and* $\partial F/\partial x|_{(x_0,\lambda_0)} = 0$ *and if through* (x_0, λ_0) *pass two and only two braches of the equilibrium curve which have both distinct tangents at* (x_0, λ_0), *then* (x_0, λ_0) *is called a* *transcritical bifurcation*.

Bifurcation diagram: 5



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2) *i)* We compute

$$V(x) = ge^{-x}\cos x$$
$$V'(x) = -ge^{-x}(\sin x + \cos x)$$
$$V''(x) = 2ge^{-x}\sin x$$

The stationary points are obtained from $V'(x) = 0$. Therefore $\sin x + \cos x = \sqrt{2}\sin(\pi/4 + x) = 0$. This means the stationary points are at 4

$$x^{(n)} = -\frac{\pi}{4} + n\pi.$$

Next we compute

$$V''(x^{(n)}) = 2ge^{\frac{\pi}{4}-n\pi} \sin\left(-\frac{\pi}{4} + n\pi\right) = \sqrt{2}(-1)^{n+1} g e^{\frac{\pi}{4}-n\pi}.$$

for $g \in \mathbb{R}^+, n$ even: $V''(x^{(n)}) < 0 \Rightarrow$maximum at $x^{(n)} \Rightarrow$saddle point at $(x^{(n)}, 0)$,
for $g \in \mathbb{R}^+, n$ odd: $V''(x^{(n)}) > 0 \Rightarrow$minimum at $x^{(n)} \Rightarrow$centre at $(x^{(n)}, 0)$,
for $g \in \mathbb{R}^-, n$ even: $V''(x^{(n)}) > 0 \Rightarrow$minimum at $x^{(n)} \Rightarrow$centre at $(x^{(n)}, 0)$,
for $g \in \mathbb{R}^-, n$ odd: $V''(x^{(n)}) < 0 \Rightarrow$maximum at $x^{(n)} \Rightarrow$saddle point at $(x^{(n)}, 0)$. 2

ii) The separatrix passes through the saddle point, i.e.

$$H(-\pi/4, 0) = 1/2 e^{\pi/4} \cos(-\pi/4) = E_{\text{saddle}}$$

Therefore the equation of the separatrix results from

$$E_{\text{saddle}} = \frac{1}{2}x_2^2 + \frac{1}{2}e^{-x_1} \cos x_1$$

that is

$$\begin{aligned} x_2 &= \pm 2\sqrt{E_{\text{saddle}} - \frac{1}{2}e^{-x_1}\cos x_1} \\ &= \pm\sqrt{\frac{1}{\sqrt{2}}e^{\pi/4} - e^{-x_1}\cos x_1} \end{aligned}$$

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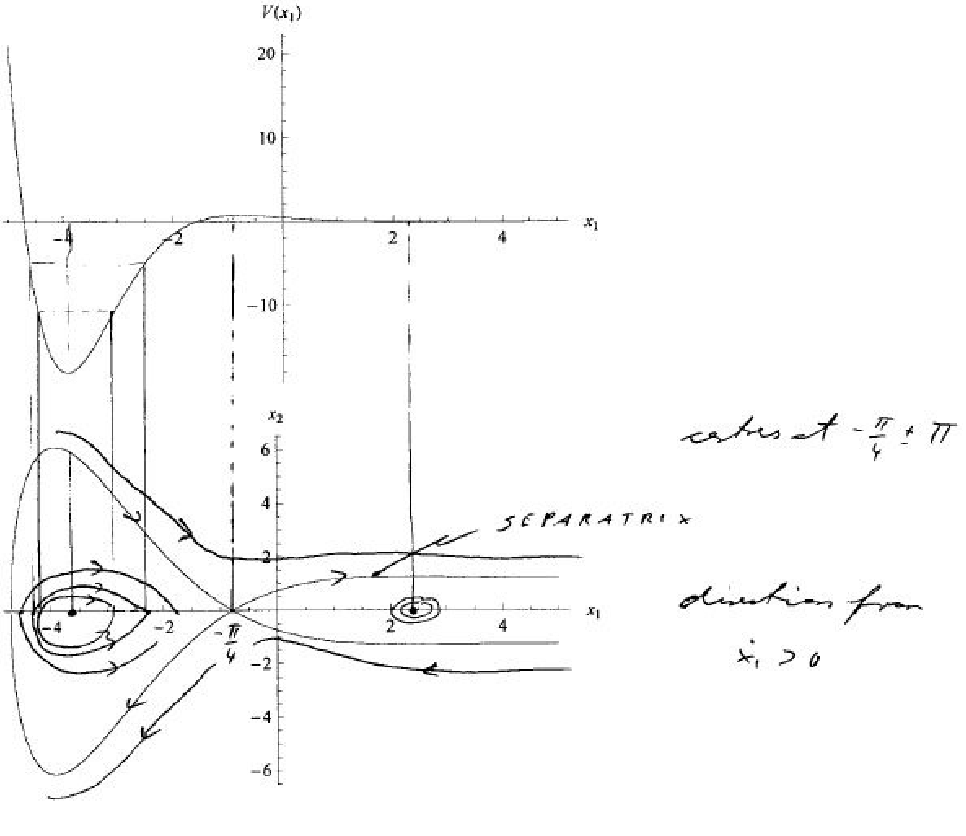

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iii) First compute the energy using the initial conditions

$$H(0,1/2) = \frac{1}{2}\left(\frac{1}{2}\right)^2 = \frac{1}{8} = E$$

The turning points result from $x_2 = 0$

$$E = \frac{1}{8} = H(x_t, 0) = \frac{1}{8}x_t^8 \quad \Rightarrow x_t = \pm 1.$$

The period is then computed from

$$T = 2\int_\alpha^\beta \frac{dx}{\sqrt{2[E - V(x)]}} = 2\int_{-1}^1 \frac{dx}{\sqrt{2[1/8 - x^8/8]}}$$
$$= 8\int_0^1 \frac{dx}{\sqrt{1-x^8}} = 8\sqrt{\pi}\Gamma(9/8)/\Gamma(5/8)$$

5

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3) We have

$$x_{n+1} = F(x_n) = \lambda x_n(4 - x_n) \qquad \text{for } \lambda \in \mathbb{R}^+.$$

i) In order to find the fixed points we need to solve

$$x = F(x) = \lambda x(4-x) \quad \Leftrightarrow \quad x(x + \frac{1}{\lambda} - 4) = 0$$

Therefore, the fixed points are 1

$$x_f^{(1)} = 0 \qquad \text{and} \qquad x_f^{(2)} = 4 - \frac{1}{\lambda}.$$

A fixed point is stable iff $|F'(x)| < 1$.

$$\Rightarrow \quad F'(x) = 4\lambda - 2\lambda x$$

$\Rightarrow \left|F'(x_f^{(1)})\right| = |4\lambda| > 1$ for $\lambda > 1/4$. $\Rightarrow x_f^{(1)}$ is unstable for $\lambda > 1/4$.

$\Rightarrow \left|F'(x_f^{(1)})\right| = |4\lambda - 8\lambda + 2| < 1$ for $-1 < 2 - 4\lambda < 1 \Rightarrow 1/4 < \lambda < 3/4$

$\Rightarrow x_f^{(2)}$ is stable for $1/4 < \lambda < 3/4$. 4

ii) A two cycle exits iff $F(F(x)) = x$

$$\Rightarrow x = 4\lambda F(x) - \lambda F^2(x)$$
$$\Rightarrow x = 16x\lambda^2 - 4x^2\lambda^2 - 16x^2\lambda^3 + 8x^3\lambda^3 - x^4\lambda^3$$

We can factorize this equation, because the fixed point is a solution of $F(F(x)) = x$. By polynomial division (or verify by multiplication)

$$[F(x) - x]\left(x^2\lambda^2 - 4x\lambda^2 - x\lambda + 4\lambda + 1\right) = 16x\lambda^2 - x - (4\lambda^2 + 16\lambda^3)x^2 + 8x^3\lambda^3 - x^4\lambda^3\,.$$

This means the existence of the two cycle is governed by the equations

$$x^2\lambda^2 - 4x\lambda^2 - x\lambda + 4\lambda + 1 = 0.$$

We solve this by 5

$$x_\pm = \frac{4\lambda^2 + \lambda \pm \lambda\sqrt{16\lambda^2 - 8\lambda - 3}}{2\lambda^2}.$$

Thus $x_\pm \in \mathbb{R}$ iff $16\lambda^2 - 8\lambda - 3 \geq 0$. Therefore the existence of the two cycle requires

$$(1 + 4\lambda)(4\lambda - 3) \geq 0 \quad \Rightarrow \quad \lambda \geq 3/4$$

Compute the solution of this equation and use it to argue that the existence of a 2-cycle requires $\lambda \geq 3/4$. 2

iii) A two cycle is stable iff for $G(x) = F(F(x))$

$$|G'(x)| < 1 \qquad \Leftrightarrow \qquad |F(x_+)F(x_-)| < 1$$

Therefore

$$\begin{aligned} |F(x_+)F(x_-)| &= |(4\lambda - 2\lambda x_+)(4\lambda - 2\lambda x_-)| \\ &= \left|\left(-1 + \sqrt{16\lambda^2 - 8\lambda - 3}\right)\left(-1 - \sqrt{16\lambda^2 - 8\lambda - 3}\right)\right| \\ &= \left|1 - \left(16\lambda^2 - 8\lambda - 3\right)\right| \\ &= \left|16\lambda^2 - 8\lambda - 4\right| < 1, \end{aligned}$$

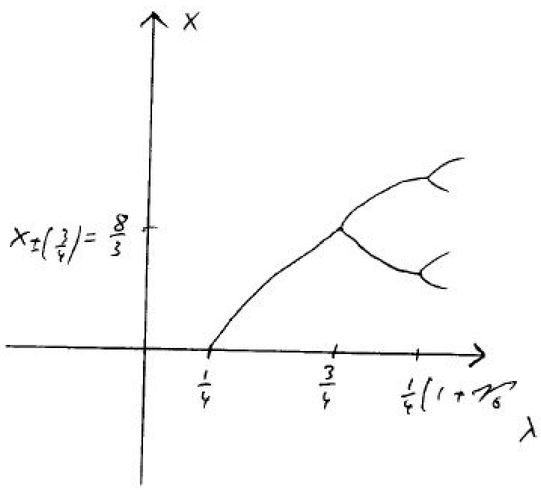
such that

$$\begin{aligned} 16\lambda^2 - 8\lambda - 5 < 0 \quad &\wedge \quad 16\lambda^2 - 8\lambda - 3 > 0 \\ \left[\lambda - \tfrac{1}{4}(1 - \sqrt{6})\right]\left[\lambda - \tfrac{1}{4}(1 + \sqrt{6})\right] < 0 \quad &\wedge \quad \left(\lambda + \tfrac{1}{4}\right)\left(\lambda - \tfrac{3}{4}\right) > 0 \\ \lambda < \tfrac{1}{4}(1 + \sqrt{6}) \quad &\wedge \quad \lambda > \tfrac{3}{4} \end{aligned}$$

$\Rightarrow$ The domain of stability for the two cycle is 6

$$\frac{3}{4} < \lambda < \frac{1}{4}(1 + \sqrt{6}).$$

The bifurcation diagram is 2



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