Dynamical Systems II

Solutions and marking scheme for coursework 2

INSTRUCTIONS: Each question carries 20 marks.

1. (i) *Bifurcation theory* investigates how the number of steady solutions of systems of the type $\dot{x} = F(x, \lambda)$ depend on the parameter λ. A *bifurcation* occurs if the solution of $\dot{x} = F(x, \lambda)$ changes its qualitative behaviour as the parameter λ varies. Considering $F(x, \lambda) = 0$ leads to a plot in the (x, λ)-plane called the *bifurcation diagram*. [2]

The fixed points are found from [2]

$$F(x, \lambda) = x^3 + \gamma x^2 - \lambda x = 0.$$

i.e. they are at the three curves

$$x_1 = 0 \qquad x_{2/3} = \frac{1}{2}\left(-\gamma \pm \sqrt{\gamma^2 + 4\lambda}\right).$$

In order to characterize the types of bifurcations we need

$$\frac{\partial F(x, \lambda)}{\partial x} = 3x^2 + 2\gamma x - \lambda \qquad \text{and} \qquad \frac{\partial F(x, \lambda)}{\partial \lambda} = -x.$$

- *A point (x_0, λ_0) on the equilibrium curve for the system $\dot{x} = F(x, \lambda)$ is called a pitchfork bifurcation if $\partial F/\partial \lambda|_{(x_0,\lambda_0)} = 0$, $\partial F/\partial x|_{(x_0,\lambda_0)} = 0$ and $d\lambda/dx$ changes sign on one branch of the equilibrium curve with distinct tangents, where $\lambda(x)$ is the solution of the equation $F(x, \lambda) = 0$.* [2]
  $\partial F/\partial \lambda|_{(x_0,\lambda_0)} = 0$ gives $x_0 = 0$ and subsequently $\partial F/\partial x|_{(x_0,\lambda_0)} = 0$ gives $\lambda_0 = 0$. Since $d\lambda/dx = 2x$ changes sign at $x_0 = 0$ and this branch has a different tangent than $x_1 = 0$, the point $(x_0, \lambda_0) = (0, 0)$ constitutes a pitchfork bifurcation.
- *A point (x_0, λ_0) on the equilibrium curve for the system $\dot{x} = F(x, \lambda)$ is called a transcritical bifurcation if $\partial F/\partial \lambda|_{(x_0,\lambda_0)} = 0$, $\partial F/\partial x|_{(x_0,\lambda_0)} = 0$ and in addition two and only two branches of the equilibrium curve pass through this point which have both distinct tangents at (x_0, λ_0).* [2]
  For $\gamma \neq 0$ we have $d\lambda/dx = 2x + \gamma$, which no longer changes sign at $x_0 = 0$. However, only two branches pass through this point and their tangents are distinct, such that $(x_0, \lambda_0) = (0, 0)$ constitutes a transcritical bifurcation.

- *A point* (x_0, λ_0) *on the equilibrium curve for the system* $\dot{x} = F(x, \lambda)$ *is called a* *turning point* *if* $\partial F/\partial \lambda|_{(x_0,\lambda_0)} \neq 0$ *and* $\partial \lambda/\partial x$ *changes sign at this point.* **4**

  From
$$\frac{\partial x_2}{\partial \lambda} = \frac{1}{\sqrt{\gamma^2 + 4\gamma}} \qquad \text{and} \qquad \frac{\partial x_3}{\partial \lambda} = -\frac{1}{\sqrt{\gamma^2 + 4\gamma}}.$$
  follows that $\partial x/\partial \lambda$ changes sign for $\lambda_0 = -\gamma^2/4$, such that $x_2(\lambda_0) = x_3(\lambda_0) = x_0 = -\gamma/2$. Since $\partial F/\partial \lambda|_{(x_0,\lambda_0)} = \gamma/2 \neq 0$ this mean
$$(x_0, \lambda_0) = (-\gamma/2, -\gamma^2/4)$$
  is a turning point for the above system.

(*ii*) We make use of the follwoing corollary: *Suppose that for the system* $\dot{\vec{x}} = \vec{F}(\vec{x})$,*we have transformed the linearized system* $\dot{\vec{x}} = A\vec{x}$,*with the help of* $\vec{x} = U\vec{y}$ *into the Jordan normal form* **4**

$$\dot{\vec{y}} = U^{-1}AU\vec{y} = J\vec{y} = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}, \tag{1}$$

with $\omega \in \mathbb{R}^+$. *Then the origin is asymptotically stable if the stability index* I, *computed from the transformed system* $\dot{\vec{y}} = \vec{Y}(\vec{y})$, *is negative.*
Thus we compute the Jacobian matrix for the system

$$\begin{aligned} \dot{x}_1 &= 9x_2 + 3x_1^2 \\ \dot{x}_2 &= \lambda x_2 - 2x_1^2 x_2 - 9x_1 - 2x_1^3 + \alpha x_1^2 \end{aligned}$$

to
$$A(\vec{x}_f) = \begin{pmatrix} 0 & 9 \\ -9 & \lambda \end{pmatrix}$$

We note that for $\lambda = 0$ this is already in Jordan normal form, such that $A = J$ and $\vec{X} = \vec{Y}$. Therefore $\omega = 9$. The only nonvanishing terms in I are

$$Y^2_{112} = -4, \quad Y^2_{11} = 2\alpha \quad \text{and} \quad Y^1_{11} = 6.$$

Therefore
$$I = \omega Y^2_{112} + Y^2_{11} Y^1_{11} = 9(-4) + 12\alpha = 12\alpha - 36.$$

This means I is negative for $\alpha < 3$, i.e. the origin is asymptotically stable for $\alpha < 3$.

(*iii*) **Hopf bifurcation theorem:** *Let the point (0,0,*λ*), with* $\lambda \in \mathbb{R}$, *be a fixed point for the system* **4**

$$\begin{aligned} \dot{x}_1 &= F_1(x_1, x_2, \lambda), & (2) \\ \dot{x}_2 &= F_2(x_1, x_2, \lambda), & (3) \end{aligned}$$

for all values of λ. *If for a particular value of* λ, *say* $\lambda = \tilde{\lambda}$,

i) *the eigenvalues* $e_1(\lambda)$ *and* $e_2(\lambda)$ *of the linearized system are purely imaginary, i.e.* $e_1(\tilde{\lambda}) \in iR$ *and* $e_2(\tilde{\lambda}) \in iR$,

ii) *the real part of the eigenvalues* $\mathrm{Re}(e_1(\lambda)) = \mathrm{Re}(e_2(\lambda))$ *satisfies*

$$\frac{d}{d\lambda} \mathrm{Re}(e_{1/2}(\lambda))\Big|_{\lambda=\tilde{\lambda}} > 0, \tag{4}$$

iii) *the origin is asymptotically stable for* $\lambda = \tilde{\lambda}$,

then the following statements hold:

a) *The point with* $\lambda = \tilde{\lambda}$ *is a bifurcation point of the system.*

b) *For* $\lambda \in (\lambda_1, \tilde{\lambda})$ *with some* $\lambda_1 < \tilde{\lambda}$ *the origin is a stable focus.*

c) *For* $\lambda \in (\tilde{\lambda}, \lambda_2)$ *with some* $\lambda_2 > \tilde{\lambda}$ *the origin is an unstable focus surrounded by a stable limit cycle whose size increases with* λ.

The eigenvalues for the Jacobian matrix with $\lambda \neq 0$ are computed to

$$e_\pm = \frac{1}{2}\left(\lambda \pm \sqrt{\lambda^2 - 324}\right).$$

i) for $\lambda = 0$ the eigenvales are purely imaginary: $e_\pm = \pm i9$.

ii) we compute

$$\frac{d}{d\lambda} \mathrm{Re}(e_{1/2}(\lambda))\Big|_{\lambda=\tilde{\lambda}=0} = \frac{1}{2} > 0.$$

iii) from part (*ii*) of the question we know that the origin is asymptotically stable for $\alpha = 2$.
Therefore the Hopf bifurcation theorem applies for $\alpha = 2$. The situation is inconclusive for $\alpha = 4$. $\boxed{\sum = 20}$

2. (*i*) **Def.:** *A system of differential equations on* $\mathbb{R}^2$ *is said to be a* <u>Hamiltonian system</u> *with one degree of freedom if there exists a twice continuously differentiable function* $H(x_1, x_2)$ *such that* $\boxed{2}$

$$\dot{x}_1 = \frac{\partial H}{\partial x_2} \qquad \text{and} \qquad \dot{x}_2 = -\frac{\partial H}{\partial x_1}. \tag{5}$$

The equations (5) are said to be the *equations of motions* correponding to the Hamiltonian H. When H does not depend explicitly on the time t, i.e. it is of the form $H(x_1(t), x_2(t))$ and not $H(x_1(t), x_2(t), t)$, the system is called *autonomous.*

(*ii*) *A dynamical system* $\boxed{2}$

$$\dot{x}_1 = F_1(x_1, x_2) \qquad \text{and} \qquad \dot{x}_2 = F_2(x_1, x_2),$$

is a Hamiltonian system if and only if

$$\mathrm{div}\, \vec{F} = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} = 0.$$

We compute

$$\operatorname{div}\vec{F} = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} = 3\mu x_1 x_2^2 + 2 - 6x_1 x_2^2 - 2 = 0.$$

Therefore the system is a Hamiltonian system when $\mu = 2$. [Sorry, there was a typo on the question sheet. It should have read $\dot{x}_2 = -\mu x_1 x_2^3 - 2x_2 + \sin(x_1^5)$ instead of $\dot{x}_2 = -\mu x_1^2 x_2^3 - 2x_2 + \sin(x_1^5)$. Full marks were therefore usually given even for answers like $\mu = 2/x_1$.]

(*iii*) **Def.:** *A Hamiltonian system which is of the form* 5

$$H(x_1, x_2) = \frac{1}{2}x_2^2 + V(x_1),$$

where $V(x_1)$ is a function which only depends on x_1 and not x_2 is called a potential system *with* potential (function) *$V(x_1)$.*

From the definition in (i) follows

$$\dot{x}_1 = \frac{\partial H}{\partial x_2} = x_2 \Rightarrow H(x_1, x_2) = \frac{1}{2}x_2^2 + f(x_1)$$
$$\dot{x}_2 = -\frac{\partial H}{\partial x_1} = -2x_1 + \frac{20x_1}{1+x_1^2} \Rightarrow H(x_1, x_2) = x_1^2 - 10\ln(1+x_1^2) + f(x_2).$$

Therefore

$$H(x_1, x_2) = \frac{1}{2}x_2^2 + x_1^2 - 10\ln(1+x_1^2) + c,$$

such that the potential is

$$V(x_1) = x_1^2 - 10\ln(1+x_1^2) + c.$$

From $V(0) = 0$ follows $c = 0$.

(*iv*) *The fixed points for the Hamiltonian system described by* 5

$$H(x_1, x_2) = \frac{1}{2}x_2^2 + V(x_1) \qquad (6)$$

are located at the points $(a_k, 0)$ with $k = 1, 2, 3, \ldots$, where the a_k are stationary points of the potential $V(x_1)$. If $V(a_k)$ is a minimum then the point $(a_k, 0)$ is a centre and if on the other hand $V(a_k)$ is a maximum the point $(a_k, 0)$ is a saddle point.

We compute the stationary points from

$$V'(x_1) = 2x_1 - \frac{20x_1}{1+x_1^2} = \frac{2x_1(x_1^2 - 9)}{1+x_1^2} = 0 \qquad \text{for } x_1 = 0, \pm 3.$$

Furthermore

$$V''(x_1) = 2 - 10\left(-\frac{4x_1^2}{(1+x_1^2)^2} + \frac{2}{1+x_1^2}\right)$$

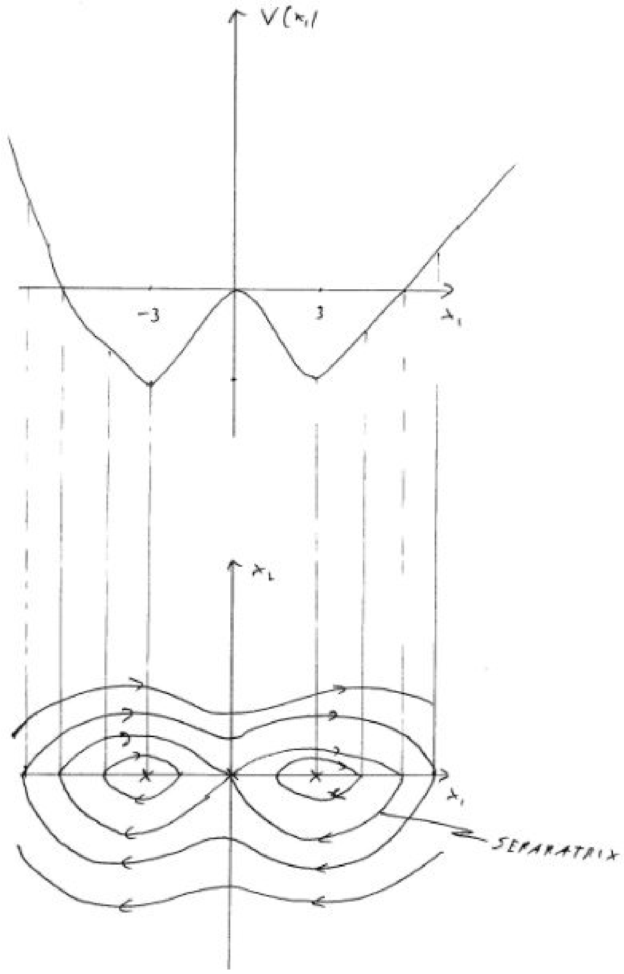
and therefore

$$V''(0) = -18 \Rightarrow x_1 = 0 \text{ is a maximum of } V(x_1) \Rightarrow (0,0) \text{ is a saddle point,}$$
$$V''(\pm 3) = \frac{18}{5} \Rightarrow x_1 = \pm 3 \text{ are minima of } V(x_1) \Rightarrow (\pm 3, 0) \text{ are centres.}$$

(v) The separatrix crosses the saddle point, i.e. $H(0,0) = 0$ is conserved on the separatrix. The equation for the separatrix is therefore 6

$$0 = \frac{1}{2}x_2^2 + x_1^2 - 10\ln(1 + x_1^2) \Rightarrow x_2 = \pm\sqrt{-2x_1^2 + 20\ln(1 + x_1^2)}.$$

The direction of time follows from $\dot{x}_1 > 0$ for $x_2 > 0$ and $\dot{x}_1 < 0$ for $x_2 < 0$.
All trajectories are bounded.
We assemble all the information in the diagram:



$\sum = 20$

3. (i) The fixed points are found from 1

$$F(x) = x \qquad \Leftrightarrow \qquad 8\lambda x - 4\lambda x^2 = x$$

This means we have fixed points at 3

$$x_f^{(1)} = 0 \qquad \text{and} \qquad x_f^{(2)} = 2 - \frac{1}{4\lambda}.$$

A fixed point x_f is stable iff $|F'(x_f)| < 1$. With $F'(x) = 8\lambda - 8\lambda x$ follows that
$x_f^{(1)}$ is stable for $|8\lambda| < 1$, that is $\lambda < 1/8$.
$x_f^{(2)}$ is stable for $|2 - 8\lambda| < 1$, that is $1/8 < \lambda < 3/8$.

(*ii*) A 2-cycle exists if $F(F(x)) = x$. Compute 6

$$\begin{aligned} x &= 8\lambda F(x) - 4\lambda F^2(x) \\ &= 8\lambda(8\lambda x - 4\lambda x^2) - 4\lambda(8\lambda x - 4\lambda x^2)^2 \\ &= 64\lambda^2 x - 64\lambda^3 x^4 + 256\lambda^3 x^3 - 256\lambda^3 x^2 - 32\lambda^2 x^2 \\ &= 32(2\lambda^2 x - 2\lambda^3 x^4 + 8\lambda^3 x^3 - 8\lambda^3 x^2 - \lambda^2 x^2) \end{aligned}$$

Since the fixed point is a solution of this equation, we can factor out the term $F(x) - x$. Not knowing the answer the can be done by polynomial devision, but in this case it is sufficient to verify that: 4

$$\begin{aligned} &(F(x) - x)\,(1 + 8\lambda - 4x\lambda - 32x\lambda^2 + 16x^2\lambda^2) \\ &= 32(2\lambda^2 x - 2\lambda^3 x^4 + 8\lambda^3 x^3 - 8\lambda^3 x^2 - \lambda^2 x^2) - x = 0 \end{aligned}$$

This means we require 3

$$1 + 8\lambda - 4x\lambda - 32x\lambda^2 + 16x^2\lambda^2 = 0$$

for a two cycle to exist. Solving this quadratic equation gives

$$x_\pm = 1 + \frac{1}{8\lambda} \pm \frac{1}{8\lambda}\sqrt{64\lambda^2 - 16\lambda - 3}$$

For this to be real we require

$$64\lambda^2 - 16\lambda - 3 \geq 0.$$

Therefore the existence of a two cycle is ensured iff

$$(8\lambda + 1)(8\lambda - 1) \geq 0,$$

which means $\lambda \geq 3/8$.

(*iii*) The 2 cycle is stable for $G(x) = F(F(x))$ 3

$$\left|G'(x)\right| < 1 \qquad \Leftrightarrow \qquad \left|F'(x_+)F'(x_-)\right| < 1$$

Compute therefore

$$|(8\lambda - 8\lambda x_+)(8\lambda - 8\lambda x_-)| = \left|4 + 16\lambda - 64\lambda^2\right| < 1$$

This means the two cycle is stable in the regime

$$\frac{3}{8} < \lambda < \frac{1}{8}(1 + \sqrt{6})$$

and unstable for $\lambda > (1 + \sqrt{6})/8$ $\sum = 20$