

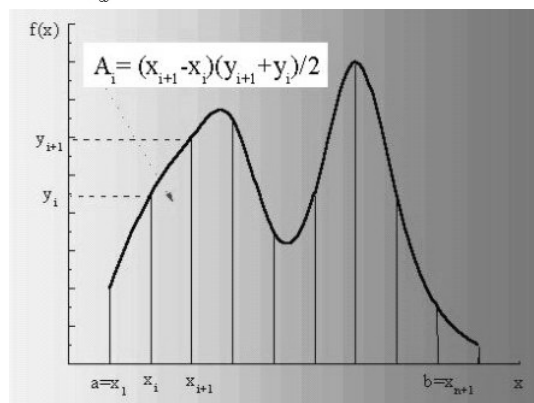
## Numerical Methods with Excel/VBA:

- Many problems in Mathematics, Physics, Economics, etc can only be solved in very idealized situations in an exact analytical fashion. Even solvable problems can often only be tackled with great effort.
- Numerical methods often lead to solutions which are extremely close to the correct answers. They lead very often quickly to some insights.
- Especially with the advance in computer technology, in terms of speed and storage capacity the limits of what can be computed are permanently pushed.
- Here we only have a glimpse at some methods to get an idea what could be possible and to apply the programming structures we have learned so far.

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### ► Numerical Integration

- Recall: 
$$I = \int_a^b f(x) dx = \text{area below the curve } f(x)$$



- Idea: approximate the integral by sums over trapezoidal areas :

$$I \approx \sum_{i=1}^n A_i$$

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- Take the subdivision of the domain [a,b] to be evenly spaced:

$$x_{i+1} - x_i = \frac{b-a}{n} = \Delta$$

⇒ Trapezoid rule for integration:

$$I \approx \Delta \left[ \frac{1}{2}(y_1 + y_{n+1}) + \sum_{i=2}^n y_i \right]$$

• Let us write a module (program) for this:

- Input: a ≡ lower integration limit

b ≡ upper integration limit

n ≡ number of subdivisions

some function f(x) which we want to integrate

- Output: approximate value for the integral

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```
Sub Nint()
```

```
  a = 0
```

```
  b = 5
```

```
  n = 100
```

```
  h = (b - a) / n
```

```
  I = h * (f(a) + f(b)) / 2
```

```
  For m = 2 To n
```

```
    I = I + f(a + h * (m - 1)) * h
```

```
  Next
```

```
  Range("B3").Value = I
```

```
End Sub
```

```
Function f(x)
```

```
  f = x ^ 4
```

```
End Function
```

The screenshot shows the VBA code editor for a module named 'Nint'. The code is as follows:

```
Sub Nint()
  a = 0
  b = 5
  n = 100
  h = (b - a) / n
  I = h * (f(a) + f(b)) / 2
  For m = 2 To n
    I = I + f(a + h * (m - 1)) * h
  Next
  Range("B3").Value = I
End Sub

Function f(x)
  f = x ^ 4
End Function
```

Put the result onto the Excel sheet into the cell B3

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- Example 1:  $\int_0^5 x^4 dx = \frac{x^5}{5} \Big|_0^5 = 625$ 
  - The program gives:
 

$n = 10$	$I = 635.4063$
$n = 100$	$I = 625.1042$
$n = 1000$	$I = 625.0010$
$n = 10000$	$I = 625.0000104$

- Example 2:  $\int_0^\pi \sin(x) dx = -\cos(x) \Big|_0^\pi = 2$ 
  - Generate the  $\pi$  by  $4 \text{ Arctan}(1)$ . In VBA this is written as  $4 * \text{Atn}(1)$ .
  - The program yields:
 

$n = 10$	$I = 1.9835$
$n = 100$	$I = 1.999836$
$n = 1000$	$I = 1.999998$

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- So far we could have solved the integrals also analytically, but not the next integral.

- Example 3:  $\int \frac{\sin(x)}{x} dx = \text{Sinus Integral function}$ 

$$\int_0^\pi \frac{\sin(x)}{x} dx \approx 1.85194$$

- How do we deal with the lower bound  $a=0$ ? This is well defined analytically, but the computer can not handle  $0/0$ , if we don't specify how to do that. Recipe: Just take the lower bound  $a$  to be a very small number, e.g.  $a=0.0000001$ .

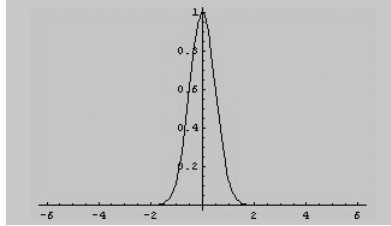
- The program yields:
 

$n = 10$	$I = 1.8493$
$n = 100$	$I = 1.851911$
$n = 1000$	$I = 1.851937$

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• Example 4:  $\int_{-\infty}^{\infty} \exp(-2x^2) dx = \frac{1}{2} \sqrt{\frac{\pi}{2}} \text{Errorfunction}(x\sqrt{2})$   
 $\int_{-\infty}^{\infty} \exp(-2x^2) dx = \sqrt{\frac{\pi}{2}} \approx 1.25331$

- How do we deal with infinity? Introduce a cut-off at some value large enough such that the mistake is small. This is possible because the integrant falls off sharply after certain values:



- Compute instead:  $\int_{-5}^5 \exp(-2x^2) dx$
- The program gives:  $n = 10 \quad I = 1.27134$   
 $n = 100 \quad I = 1.253314$   
 $n = 1000 \quad I = 1.2533$

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- Different types of methods:

- Simpson's 1/3 rule (based on three adjacent points):

$$I = \int_a^b f(x) dx \approx \frac{\Delta}{3} \left[ \sum_{i=1,3,5,\dots}^{n-2} y_i + 4y_{i+1} + y_{i+2} \right]$$

- Simpson's 3/8 rule (based on four adjacent points):

$$I = \int_a^b f(x) dx \approx \frac{3}{8} \Delta \left[ \sum_{i=1,4,7,\dots}^{n-3} y_i + 3y_{i+1} + 3y_{i+2} + y_{i+3} \right]$$

- Runge-Kutta methods, Monte Carlo integration,...
- Here we do not derive these rules, but just take them as facts. See a different course on numerical methods for details.
- Let us implement the Simpson's 3/8 rule as a user defined function.
- Implement the Simpson's 1/3 rule in Labsession 3.

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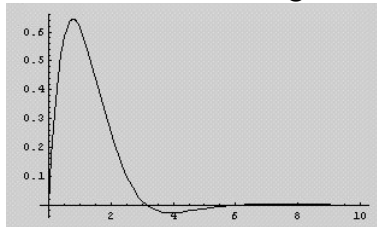
```

Function Nintff(a, b, n)
  h = (b - a) / n
  I = 0
  For m = 1 To n - 2 Step 3
    I = I + (f(a+h*(m-1)) + 3* f(a+h* m) + 3* f(a+h*(m+1))+f(a+h*(m+2)))
  Next
  Nintff = I * h * 3 / 8
End Function

```

• Example 1:  $2 \int_0^{\infty} \sin(x) \exp(-x) dx = 1$

• A problem here is to find a good cut-off for the upper limit.



⇒ b=10 ?

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- Compare different integration methods:

b=10	Trapezoid:		1/3 Simpson		3/8 Simpson
n=10	0.83920049		0.97383313		0.93930204
n=100	0.99839888		1.00006056		1.00006255
n=1000	1.00004637		1.00006279		1.00006328
n=10000	1.00006265		1.00006279		1.00006284
b=20	Trapezoid:		1/3 Simpson		3/8 Simpson
n=10	0.43524312		0.61641151		0.49069239
n=100	0.99334224		0.99996411		0.9999185
n=1000	0.99993333		0.99999999		0.99999999
n=10000	0.99999933		1		1

- In this example we introduce an additional error though the cut-off.
- When the subdivision of the interval is large enough the three methods are almost equally good.

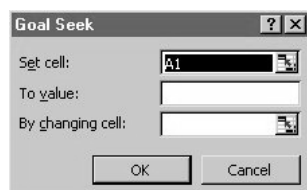
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- Limitations:
  - The speed of the computer.
  - The accuracy of the numerical method used. (In principle there exist procedures to estimate the errors made.)
  - The accuracy of the computer, i.e. in computing functions used in the program and variables employed such as single or double precision.
  - Various other approximations such as division by zero, cut-offs for lower and upper bounds etc.
- There exist different types of numerical methods for other mathematical problems, such as solving polynomial equations, solving differential equations etc.
- Some methods are implemented in Excel as Built-in functions:

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### ► Goal Seek

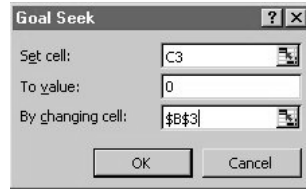
- Goal seek is a numerical routine implemented in Excel in form of a built-in function. It can be used to solve equations.
- Usage: select Tools → Goal Seek ↵ → a dialog window opens



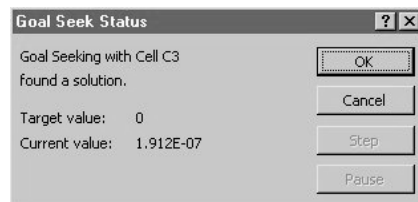
- Set cell contains the left hand side of an equation you want to solve
  - To value contains the RHS of the equation
  - By changing cell contains the variable of the equation
- Disadvantage: You have to guess a value near the answer.
  - Example: Solve the equation:  $2x^2 - 9x - 5 = 0$   
(We expect to find:  $x_1 = -1/2$  and  $x_2 = 5$ )
    - Type into the cell C3:  $=2*B3^2 - 9*B3 - 5$
    - Type into the cell C4:  $=2*B4^2 - 9*B4 - 5$

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- Type into the cell B3 some starting value, e.g. -10
- open the Goal Seek dialog box and fill in



- OK ↵ →



- The cell B3 and C3 have changed to -0.5 and 1.912E-07
- Repeat this process for the cells C4 and B3 to find the other solution. (You need a new guess for the starting value.)
- A more sophisticated method is the Excel Solver.