Solutions MA1607 Geometry & Vectors (2008)

1) Given are the vectors $\boxed{\sum = 12}$

$$\vec{u} = \lambda\vec{\imath} - 7\vec{\jmath} - \vec{k}, \qquad \text{and} \qquad \vec{v} = 2\vec{\imath} - \vec{\jmath} + 2\vec{k}.$$

(*i*) In general we have

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}|\,|\vec{v}|}.$$

We compute

$$\left.\begin{array}{r} \vec{u}\cdot\vec{v} = 2\lambda + 7 - 2 \\ |\vec{u}| = \sqrt{\lambda^2 + 49 + 1} \\ |\vec{v}| = \sqrt{4+1+4} \end{array}\right\} \Rightarrow \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} = \frac{2\lambda+5}{3\sqrt{\lambda^2+50}}.$$

Therefore

$$\frac{9}{2} = \frac{(2\lambda+5)^2}{\lambda^2+50} \Rightarrow \frac{9}{2}\lambda^2 + \frac{9}{2}50 = 4\lambda^2 + 20\lambda + 25 \Rightarrow \boxed{\lambda = 20}.$$

$\boxed{5}$

(*ii*) Take the unknown vector to be of the general form

$$\vec{w} = a\vec{\imath} + b\vec{\jmath} + c\vec{k} \qquad \text{with } a, b, c \in \mathbb{R}.$$

Since $\vec{u} \perp \vec{w}$ and $\vec{v} \perp \vec{w}$ we have

$$\left.\begin{array}{r} \vec{u}\cdot\vec{w} = -a - 7b - c = 0 \\ \vec{v}\cdot\vec{w} = 2a - b + 2c = 0 \end{array}\right\} \Rightarrow b = 0, a = -c.$$

The vector $\vec{w}$ has length $\sqrt{90}$

$$\vec{w}\cdot\vec{w} = 90 = a^2 + b^2 + c^2 \Rightarrow 90 = a^2 + a^2 \Rightarrow a = \pm 3\sqrt{5}.$$

Therefore

$$\boxed{\vec{w} = \pm 3\sqrt{5}(\vec{\imath} - \vec{k})}.$$

$\boxed{5}$

(*iii*) We compute

$$\begin{aligned} \vec{u}\times\vec{v} &= \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 14 & -7 & -1 \\ 2 & -1 & 2 \end{vmatrix} = \begin{vmatrix} \vec{\imath}+2\vec{\jmath} & \vec{\jmath} & \vec{k} \\ 0 & -7 & -1 \\ 0 & -1 & 2 \end{vmatrix} \\ &= (\vec{\imath}+2\vec{\jmath})(-14-1) = \boxed{-15(\vec{\imath}+2\vec{\jmath})}. \end{aligned}$$

$\boxed{2}$

2) (i) We scalar multiply the original equation by $\vec{b}$ $\boxed{\sum = 12}$

$$\lambda\vec{x} + (\vec{x}\cdot\vec{b})\vec{a} = \vec{c} \qquad \Big|\cdot\vec{b} \tag{1}$$

$$\Rightarrow \quad \lambda\vec{x}\cdot\vec{b} + (\vec{x}\cdot\vec{b})\vec{a}\cdot\vec{b} = \vec{c}\cdot\vec{b} \tag{2}$$

$$\Rightarrow \quad \vec{x}\cdot\vec{b} = \frac{\vec{c}\cdot\vec{b}}{\lambda + \vec{a}\cdot\vec{b}} \qquad \text{for } \lambda + \vec{a}\cdot\vec{b} \neq 0$$

Substituting this into (1) gives

$$\lambda\vec{x} + \frac{\vec{c}\cdot\vec{b}}{\lambda + \vec{a}\cdot\vec{b}}\vec{a} = \vec{c} \Rightarrow \boxed{\vec{x} = \tfrac{1}{\lambda}\left(\vec{c} - \tfrac{\vec{c}\cdot\vec{b}}{\lambda+\vec{a}\cdot\vec{b}}\right) \qquad \text{for } \lambda + \vec{a}\cdot\vec{b} \neq 0}.$$

$\boxed{6}$

When $\lambda + \vec{a}\cdot\vec{b} = 0$ it follows from (2) that $\vec{c}\cdot\vec{b} = 0$

$$\Rightarrow \boxed{\vec{x} = \tfrac{1}{\lambda}\vec{c} + \kappa\vec{a} \qquad \text{for } \kappa\in\mathbb{R},\ \lambda + \vec{a}\cdot\vec{b} = 0}.$$

$\boxed{2}$

(ii) We cross multiply the original equation by $\vec{a}$ from the left

$$\vec{a}\times\vec{x}\times\vec{a} = \vec{a}\times\vec{b}. \tag{3}$$

Using the general identity

$$\vec{u}\times\vec{v}\times\vec{w} = (\vec{u}\cdot\vec{w})\vec{v} - (\vec{u}\cdot\vec{v})\vec{w}$$

we can re-write (3) as

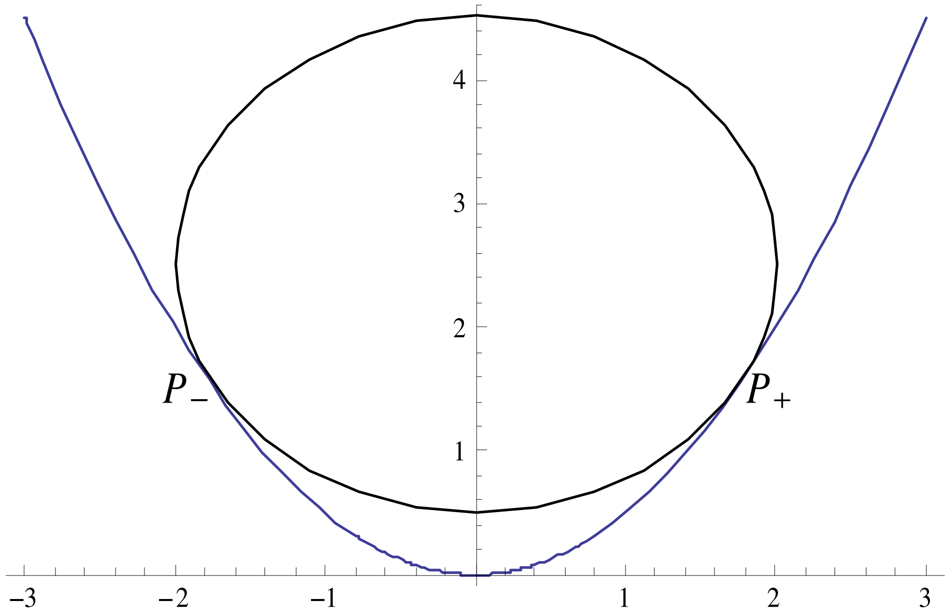
$$(\vec{a}\cdot\vec{a})\vec{x} - (\vec{a}\cdot\vec{x})\vec{a} = \vec{a}\times\vec{b}.$$

Comparing with (1), we identify $\lambda = \vec{a}\cdot\vec{a}$ and $\vec{b} = -\vec{a}$, such that $\lambda + \vec{a}\cdot\vec{b} = 0$. The solution is therefore

$$\boxed{\vec{x} = \tfrac{1}{\vec{a}\cdot\vec{a}}\vec{a}\times\vec{b} + \kappa\vec{a} \qquad \text{for } \kappa\in\mathbb{R}}.$$

$\boxed{4}$

3) (i) $\boxed{\sum = 12}$



$\boxed{3}$

(ii) The equation of the parabola is

$$y = \frac{1}{2}x^2$$

and the equation of the circle is

$$x^2 + (y-a)^2 = 4.$$

Differentiating both equations gives

$$\frac{dy}{dx} = x \qquad \text{and} \qquad 2x + 2(y-a)\frac{dy}{dx} = 0.$$

Since the tangents are the same

$$\Rightarrow 1+(y-a) = 0 \qquad \Rightarrow (y-a) = -1 \qquad \Rightarrow x^2+1 = 4 \qquad \Rightarrow x = \pm\sqrt{3}, y = \frac{3}{2}$$

The points of intersection are $\boxed{P_\pm = (\pm\sqrt{3}, 3/2)}$. $\boxed{7}$

The center results from $(3/2 - a) = -1$, i.e. $\boxed{(0, 5/2)}$.

The intersection with the y-axis is obtained from $(y - 5/2)^2 = 4$, i.e. $\boxed{y = 1/2, 9/2}$.

$\boxed{2}$

4) (i) With $A(6,1,3)$, $B(4,5,1) \Rightarrow \overrightarrow{AB} = -2\vec{\imath} + 4\vec{\jmath} - 2\vec{k}$ $\boxed{\sum = 12}$

$\Rightarrow$ equation of the line through A and B

$$\mathcal{L}: \ \frac{x-6}{-2} = \frac{y-1}{4} = \frac{z-3}{-2} = \lambda$$

$\Rightarrow P(6-2\lambda, 1+4\lambda, 3-2\lambda) \in \mathcal{L}$

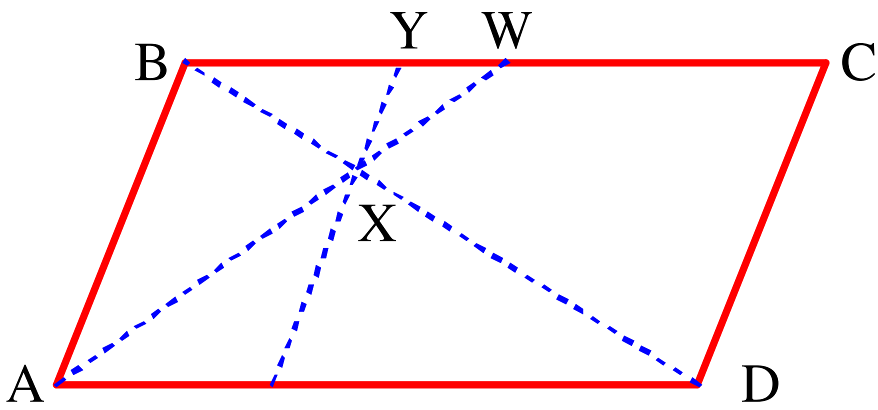
$\Rightarrow P \in yz$-plane $\Rightarrow x = 0 \Rightarrow \lambda = 3 \ \Rightarrow \ \boxed{P(0, 13, -3)}$. $\boxed{6}$

(*ii*) $\mathcal{L}$ intersects $\mathcal{P}$ for

$$\begin{aligned} 2(6-2\lambda)+(1+4\lambda)-3(3-2\lambda) &= 16 \\ 4+6\lambda &= 16 \Rightarrow \lambda = 2 \end{aligned}$$

$\Rightarrow \boxed{P(2,9,-1) = \mathcal{L} \cap \mathcal{P}}$. $\boxed{6}$

5) (*i*) $\boxed{\sum = 12}$



$\boxed{3}$

(*ii*) Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two parallel lines in a plane $\mathcal{P}$. $\mathcal{M}$ and $\mathcal{N}$ are two different lines in the same plane crossing $\mathcal{L}_1$ and $\mathcal{L}_2$ in the points M_1, N_1, M_2, N_2 and intersect in the point X. Then

$$XM_1 : XM_2 = XN_1 : XN_2.$$

$\boxed{3}$

(*iii*) Using the similarity axiom we read off the figure

$$\frac{BY}{YC} = \frac{BX}{XD}, \qquad \frac{WX}{XA} = \frac{WY}{YB}, \qquad \frac{BX}{XD} = \frac{XW}{AX}$$

and

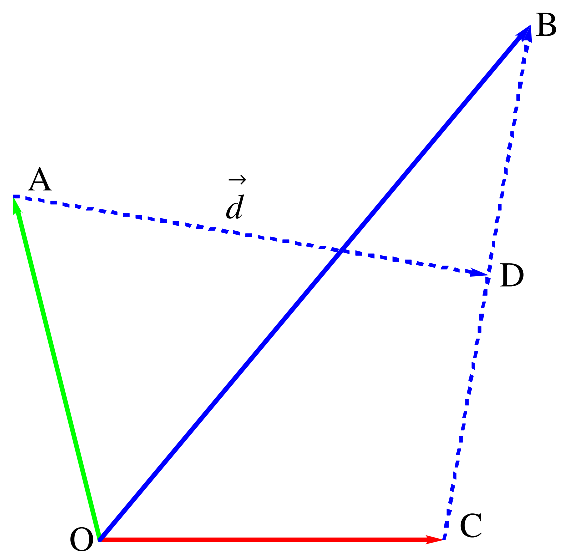
$$\frac{1}{2} = \frac{BW}{BC} = \frac{BY+YW}{BY+YC} = \frac{1+YW/BY}{1+YC/BY} = \frac{1+WX/XA}{1+XD/BX} = \frac{1+BX/XD}{1+XD/BX} = \frac{BX}{XD}.$$

Therefore

$$\frac{DX}{XB} = 2.$$

$\boxed{6}$

6) (i) $\boxed{\sum = 26}$



$\boxed{3}$

(ii) In general we have

$$|\vec{u} \times \vec{v}| = |\vec{u}|\,|\vec{v}|\sin\theta.$$

For $\theta = \pi/2$ we can use this to compute

$$\left|\vec{d} \times \overrightarrow{CD}\right| = \left|\vec{d}\right|\left|\overrightarrow{CD}\right|$$

From figure $\overrightarrow{CD} = \lambda\overrightarrow{CB} = \lambda(\vec{b} - \vec{c})$ for some $\lambda \in \mathbb{R}$. Therefore

$$\left|\vec{d}\right| = \frac{\left|\vec{d} \times \lambda(\vec{b} - \vec{c})\right|}{\left|\lambda(\vec{b} - \vec{c})\right|} = \frac{\left|\vec{d} \times (\vec{b} - \vec{c})\right|}{\left|\vec{b} - \vec{c}\right|}. \tag{4}$$

We also read off the figure

$$\vec{d} = -\vec{a} + \vec{c} + \lambda(\vec{b} - \vec{c}) \tag{5}$$

and compute

$$\begin{aligned} \vec{d} \times \vec{c} &= -\vec{a} \times \vec{c} + \vec{c} \times \vec{c} + \lambda(\vec{b} \times \vec{c} - \vec{c} \times \vec{c}) \\ \vec{d} \times \vec{b} &= -\vec{a} \times \vec{b} + \vec{c} \times \vec{b} + \lambda(\vec{b} \times \vec{b} - \vec{c} \times \vec{b}). \end{aligned}$$

With $\vec{c} \times \vec{c} = \vec{b} \times \vec{b} = 0$ we obtain

$$\begin{aligned} \vec{d} \times (\vec{b} - \vec{c}) &= -\vec{a} \times \vec{b} + \vec{c} \times \vec{b} - \lambda\vec{c} \times \vec{b} + \vec{a} \times \vec{c} - \lambda\vec{b} \times \vec{c} \\ &= -\left(\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}\right). \end{aligned}$$

Therefore with (4) follows

$$\boxed{\left|\vec{d}\right| = \frac{\left|\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}\right|}{\left|\vec{b}-\vec{c}\right|}}.$$

$\boxed{10}$

(iii) Compute

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ -\frac{1}{4} & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = -\vec{k}, \vec{b}\times\vec{c} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 1 & 0 & 0 \\ \frac{5}{4} & \frac{3}{2} & 0 \end{vmatrix} = \frac{3}{2}\vec{k}, \vec{c}\times\vec{a} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{5}{4} & \frac{3}{2} & 0 \\ -\frac{1}{4} & 1 & 0 \end{vmatrix} = \frac{13}{8}\vec{k}.$$

Therefore

$$\left.\begin{array}{r} \left|\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}\right| = \frac{17}{8} \\ \left|\vec{b}-\vec{c}\right| = \left|-\frac{1}{4}\vec{\imath}-\frac{3}{2}\vec{\jmath}\right| = \frac{\sqrt{37}}{4} \end{array}\right\} \Rightarrow \boxed{\left|\vec{d}\right| = \frac{17}{2\sqrt{37}}}.$$

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(iv) From (5)

$$\begin{aligned} \vec{d} &= -\vec{a}+\vec{c}+\lambda(\vec{b}-\vec{c}) \\ &= \frac{1}{4}\vec{\imath}-\vec{\jmath}+\frac{5}{4}\vec{\imath}+\frac{3}{2}\vec{\jmath}+\lambda\left(-\frac{1}{4}\vec{\imath}-\frac{3}{2}\vec{\jmath}\right) = \left(\frac{3}{2}-\frac{1}{4}\lambda\right)\vec{\imath}+\left(\frac{1}{2}-\frac{3}{2}\lambda\right)\vec{\jmath} \end{aligned}$$

Then

$$\begin{aligned} &\Rightarrow \vec{d}\cdot\vec{d} = \left(\frac{3}{2}-\frac{1}{4}\lambda\right)^2+\left(\frac{1}{2}-\frac{3}{2}\lambda\right)^2 = \frac{17^2}{4\cdot 37} \\ &\Rightarrow \frac{17^2}{4\cdot 37} = \frac{5}{2}-\frac{9}{4}\lambda+\frac{37}{16}\lambda^2 \Rightarrow \lambda = \frac{18}{37} \\ &\Rightarrow \overrightarrow{OD} = \vec{d}+\vec{a} = \left(\frac{3}{2}+\frac{1}{4}\frac{18}{37}\right)\vec{\imath}+\left(\frac{1}{2}-\frac{3}{2}\frac{18}{37}\right)\vec{\jmath}-\frac{1}{4}\vec{\imath}+\vec{\jmath} \\ &\Rightarrow \boxed{\overrightarrow{OD} = \frac{167}{148}\vec{\imath}+\frac{57}{74}\vec{\jmath}}. \end{aligned}$$

8

$\sum = 26$

7) (i) We have

$$\begin{aligned} \mathcal{L}_1 &: \quad \frac{x+1}{2} = y-1 = \frac{z-2}{3} = \lambda \\ \mathcal{L}_2 &: \quad -x = \frac{y+9}{3} = z+4 = \mu \end{aligned}$$

with $\lambda, \mu \in \mathbb{R}$. Therefore

$$P(2\lambda-1, \lambda+1, 3\lambda+2) \in \mathcal{L}_1 \quad \text{and} \quad Q(-\mu, 3\mu-9, 3\lambda+2) \in \mathcal{L}_2 \qquad (6)$$

For $P = Q$ we need to solve

$$\begin{aligned} 2\lambda-1 &= -\mu && (7) \\ \lambda+1 &= 3\mu-9 && (8) \\ 3\lambda+2 &= 3\lambda+2 && (9) \end{aligned}$$

Form (7) and (8) follows $\mu = 3$ and $\lambda = -1$. Equation (9) is satisfied for these values, i.e. $-1 = -1$.

$\Rightarrow$ The two lines intersect in

$$\boxed{P(-3,0,-1) = \mathcal{L}_1 \cap \mathcal{L}_2}.$$

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(ii) $\mathcal{L}_1$ is parallel to the vector $\vec{v}_1 = 2\vec{\imath} + \vec{\jmath} + 3\vec{k}$
$\mathcal{L}_2$ is parallel to the vector $\vec{v}_2 = -\vec{\imath} + 3\vec{\jmath} + \vec{k}$
$\Rightarrow \vec{v}_1 \times \vec{v}_2$ is perpendicular to $\mathcal{L}_1$ and $\mathcal{L}_2$
We compute

$$\vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 2 & 1 & 3 \\ -1 & 3 & 1 \end{vmatrix} = -8\vec{\imath} - 5\vec{\jmath} + 7\vec{k}$$

$\Rightarrow \mathcal{P}_1 : \ -8x - 5y + 7z = d$ for some $d \in \mathbb{R}$
We have $P \in \mathcal{P}_1$ for say $\lambda = 0$ in (6) $P(-1,1,2) \Rightarrow 8 - 5 + 14 = d \Rightarrow d = 17$.
$\Rightarrow$

$$\boxed{\mathcal{P}_1 : \ -8x - 5y + 7z = 17}.$$

6

(iii) Taking $P(x,y,z)$ to be an arbitrary point in the plane $\mathcal{P}_2$, the following vectors are in this plane:

$$\begin{aligned} \overrightarrow{AB} &= 2\vec{\imath} + \vec{\jmath} - \vec{k} \in \mathcal{P}_2 \\ \overrightarrow{AC} &= 3\vec{\imath} + 2\vec{\jmath} + 4\vec{k} \in \mathcal{P}_2 \\ \overrightarrow{AP} &= x\vec{\imath} + (y-3)\vec{\jmath} + (z-1)\vec{k} \in \mathcal{P}_2 \end{aligned}$$

The vector $\overrightarrow{AB} \times \overrightarrow{AC}$ is perpendicular to the plane, such that $\overrightarrow{AP} \cdot \overrightarrow{AB} \times \overrightarrow{AC} = 0$.
Compute

$$\overrightarrow{AP} \cdot \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} x & y-3 & z-1 \\ 2 & 1 & -1 \\ 3 & 2 & 4 \end{vmatrix} = 0$$

$\Rightarrow$The plane containing the points A, B, C is

$$\boxed{\mathcal{P}_2 : \ 32 + 6x - 11y + z = 0}.$$

8

(iv) A normal vector to $\mathcal{P}_1$ is $\vec{\eta}_1 = -8\vec{\imath} - 5\vec{\jmath} + 7\vec{k}$.
A normal vector to $\mathcal{P}_2$ is $\vec{\eta}_2 = 6\vec{\imath} - 11\vec{\jmath} + \vec{k}$.
$\Rightarrow \vec{\eta}_1 \times \vec{\eta}_2$ is parallel to $\mathcal{L} = \mathcal{P}_1 \cap \mathcal{P}_2$.
Compute

$$\vec{\eta}_1 \times \vec{\eta}_2 = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ -8 & -5 & 7 \\ 6 & -1 & 1 \end{vmatrix} = 72\vec{\imath} + 50\vec{\jmath} + 118\vec{k}$$

Any point on the line has to satisfy the two equations

$$\begin{aligned} 6x - 11y + z &= -32 \\ -8x - 5y + 7z &= 17. \end{aligned}$$

Taking $y = 0$ gives as solution $x = 241/50$ and $z = -77/25$.

$\Rightarrow$ The line of intersection is

$$\boxed{\mathcal{L}: \quad \frac{x-241/62}{72} = \frac{y}{50} = \frac{z+77/25}{118}} \quad .$$

Equivalently, taking $z = 0$ gives as solution $x = -347/118$ and $z = -77/59$.
$\Rightarrow$ The line of intersection is

$$\boxed{\mathcal{L}: \quad \frac{x+347/118}{72} = \frac{y-77/59}{50} = \frac{z}{118}} \quad .$$

Equivalently, taking $x = 0$ gives as solution $x = -347/118$ and $z = -77/59$.
$\Rightarrow$ The line of intersection is

$$\boxed{\mathcal{L}: \quad \frac{x}{72} = \frac{y-241/72}{50} = \frac{z-347/72}{118}} \quad .$$

$\boxed{6}$

8) (i) The distance d_1 of the centre C_1 of $\mathcal{S}_1$, i.e. the origin to the plane $\mathcal{P}$ is: $\boxed{\sum = 26}$

$$d_1 = \left| \frac{-6}{\sqrt{\mu^2 + \lambda^2 + 1}} \right|$$

For the point in $\mathcal{P}$ to be on the sphere as well we require

$$d_1 = 3 \Rightarrow 3\sqrt{\mu^2 + \lambda^2 + 1} = 6 \Rightarrow \boxed{\mu^2 + \lambda^2 = 3}.$$

$\boxed{8}$

(ii) The centre C_2 of $\mathcal{S}_2$ is $C_2(0, -6, 0)$.
The distance d_2 of C_2 to the plane $\mathcal{P}$ is:

$$d_2 = \left| \frac{-6\mu - 6}{\sqrt{\mu^2 + \lambda^2 + 1}} \right|$$

For this point to be on $\mathcal{S}_2$ as well we need $d_2 = 6$.
$\Rightarrow 6\sqrt{\mu^2 + \lambda^2 + 1} = 6(\mu + 1) \Rightarrow \mu^2 + \lambda^2 + 1 = \mu^2 + 2\mu + 1 \Rightarrow \boxed{\lambda^2 = 2\mu}.$ $\boxed{8}$

(iii) For the plane $\mathcal{P}$ to be tangent to $\mathcal{S}_1$ and $\mathcal{S}_2$ we have to solve

$$\mu^2 + \lambda^2 = 3 \qquad \wedge \qquad \lambda^2 = 2\mu.$$

$\Rightarrow \mu^2 + 2\mu - 3 = 0 \Rightarrow \mu_\pm = -1 \pm 2$
Since $\lambda^2 = 2\mu > 0$ we can discard μ_-.
$\Rightarrow \mu = 1 \Rightarrow \lambda = \pm\sqrt{2}.$
The tangent planes to $\mathcal{S}_1$ and $\mathcal{S}_2$ are therefore

$$\boxed{\mathcal{P}_\pm : \ \pm\sqrt{2}x + y + z = 6}.$$

$\boxed{8}$

(iv) For instance the points $A(0, 0, 6)$ and $B(0, 6, 0)$ are in $\mathcal{P}$.

$$\Rightarrow \overrightarrow{AB} = (0, 6, -6) \in \mathcal{P}.$$

$\boxed{2}$