Geometry & Vectors

Exercises 3

1) $\vec{e}_v$ and $\vec{e}_u$ are unit vectors in a Euclidean space. The angle between them is $\pi/6$. Find the constant λ such that the vector $\lambda\vec{e}_v - \vec{e}_u$ is perpendicular to $\vec{e}_u$.

2) The vectors $\vec{\imath}, \vec{\jmath}, \vec{k}$ constitute an orthonormal basis. Compute the projection of the vector $\vec{u} = 2\vec{\imath} + \vec{\jmath} - \vec{k}$ onto the vector $\vec{v} = 3\vec{\imath} + 10\vec{\jmath} + \vec{k}$.

3) The vectors $\vec{\imath}, \vec{\jmath}, \vec{k}$ constitute an orthonormal basis. Given are the two vectors

$$\vec{u} = 5\vec{\imath} + 2\vec{\jmath} - 4\vec{k} \qquad \text{and} \qquad \vec{v} = \frac{4}{5}\vec{\imath} - 2\vec{\jmath} + \gamma\vec{k} \qquad \text{with } \gamma \in \mathbf{R} \ .$$

Fix the constant γ such that the angle between the vectors $\vec{u}$ and $\vec{v}$ is $\pi/3$.

4) Consider the triangle OAB, where O is the origin and A=(3,6,-2) and B=(4,-1,3). Use the scalar product of the relevant position vectors to determine all three angles in the triangle OAB.

5) Derive the following trigonometric identities by using vectors only:

i) The cosine rule for the sum of two angles

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \ .$$

ii) The cosine rule for a triangle ABC

$$|\overrightarrow{AC}|^2 = |\overrightarrow{AB}|^2 + |\overrightarrow{BC}|^2 - 2|\overrightarrow{AB}||\overrightarrow{BC}|\cos\left(\widehat{ABC}\right) \ . \tag{1}$$

iii) By assuming the identity $\sin^2\varphi + \cos^2\varphi = 1$ deduce from (1) the sine rule for the triangle ABC

$$\sin\left(\widehat{ABC}\right) = \frac{2\sqrt{s(s-|\overrightarrow{AB}|)(s-|\overrightarrow{BC}|)(s-|\overrightarrow{AC}|)}}{|\overrightarrow{AB}||\overrightarrow{BC}|} \ . \tag{2}$$

where $s = \left(|\overrightarrow{AB}| + |\overrightarrow{BC}| + |\overrightarrow{AC}|\right)/2$ denotes the semiperimiter.

iv) When given all three length of the sides of a triangle, can one use both equations (1) or (2) to determine the angles uniquely?

6) ABCD constitutes a parallelogram. E is a point on the diagonal BD such that BE=2/3BD and F is the midpoint of the side DC. Show that

$$\overrightarrow{EF} = \frac{1}{6}(2\overrightarrow{AC} - \overrightarrow{AB}) \ .$$

7) Compute the work (in standard units) needed to move an object from a point $A = (-4, 3, -1)$ to a point $B(1, 11, 2)$ when applying a force $\vec{F} = 5\vec{\imath} + 2\vec{\jmath} - 4\vec{k}$.

Solutons exercises 3

1) $\lambda = 2/\sqrt{3}$

2) $\vec{u} \cdot \vec{e}_v = 15/\sqrt{115}$

3) $\gamma = -6\sqrt{29/95}$

4) $\widehat{AOB} = 90°$, $\widehat{OAB} = 36.07°$, $\widehat{OBB} = 53.93°$

5)
i) Analogous to lecture for $\cos(\alpha - \beta)$.

ii) Expand the dot product.

iii) $\sin^2\varphi + \cos^2\varphi = 1 \to \sin\varphi = \sqrt{1 - \cos^2\varphi}$. Then factorise the expression under the square root.

iv) (2) gives ambiguous results as $\sin\varphi = \sin(\pi - \varphi)$.

7) $W = 29$.