Geometry and Vectors

Coursework 2

SOLUTIONS:

$\boxed{\sum = 12}$

1. (*i*) With $A(6,1,3)$, $B(4,5,1) \Rightarrow \overrightarrow{AB} = -2\vec{\imath} + 4\vec{\jmath} - 2\vec{k}$
$\Rightarrow$ equation of the line through A and B

$$\mathcal{L}: \ \frac{x-6}{-2} = \frac{y-1}{4} = \frac{z-3}{-2} = \lambda$$

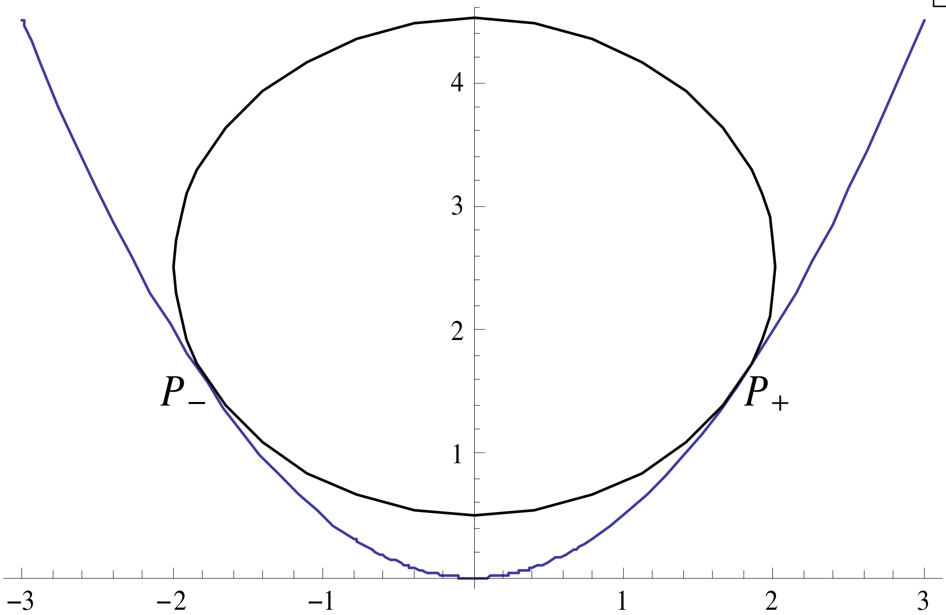
$\Rightarrow P(6-2\lambda, 1+4\lambda, 3-2\lambda) \in \mathcal{L}$
$\Rightarrow P \in yz\text{-plane} \Rightarrow x = 0 \Rightarrow \lambda = 3 \ \Rightarrow \ \boxed{P(0,13,-3)}$. $\boxed{6}$

(*ii*) $\mathcal{L}$ intersects $\mathcal{P}$ for

$$\begin{aligned} 2(6-2\lambda) + (1+4\lambda) - 3(3-2\lambda) &= 16 \\ 4 + 6\lambda &= 16 \Rightarrow \lambda = 2 \end{aligned}$$

$\Rightarrow \boxed{P(2,9,-1) = \mathcal{L} \cap \mathcal{P}}$. $\boxed{6}$

2. (*i*) $\boxed{\sum = 12}$



$\boxed{3}$

(ii) The equation of the parabola is

$$y = \frac{1}{2}x^2$$

and the equation of the circle is

$$x^2 + (y-a)^2 = 4.$$

Differentiating both equations gives

$$\frac{dy}{dx} = x \qquad \text{and} \qquad 2x + 2(y-a)\frac{dy}{dx} = 0.$$

Since the tangents are the same

$$\Rightarrow 1+(y-a) = 0 \qquad \Rightarrow (y-a) = -1 \qquad \Rightarrow x^2+1 = 4 \qquad \Rightarrow x = \pm\sqrt{3}, y = \frac{3}{2}$$

The points of intersection are $\boxed{P_\pm = (\pm\sqrt{3}, 3/2)}$. $\boxed{7}$

The center results from $(3/2 - a) = -1$, i.e. $\boxed{(0, 5/2)}$.

The intersection with the y-axis is obtained from $(y-5/2)^2 = 4$, i.e. $\boxed{y = 1/2, 9/2}$.

$\boxed{2}$

$\boxed{\sum = 26}$

3. (i) We have

$$\begin{aligned} \mathcal{L}_1 : &\quad \frac{x+1}{2} = y - 1 = \frac{z-2}{3} = \lambda \\ \mathcal{L}_2 : &\quad -x = \frac{y+9}{3} = z + 4 = \mu \end{aligned}$$

with $\lambda, \mu \in \mathbb{R}$. Therefore

$$P(2\lambda - 1, \lambda + 1, 3\lambda + 2) \in \mathcal{L}_1 \quad \text{and} \quad Q(-\mu, 3\mu - 9, 3\lambda + 2) \in \mathcal{L}_2 \qquad (1)$$

For $P = Q$ we need to solve

$$\begin{aligned} 2\lambda - 1 &= -\mu && (2) \\ \lambda + 1 &= 3\mu - 9 && (3) \\ 3\lambda + 2 &= 3\lambda + 2 && (4) \end{aligned}$$

Form (2) and (3) follows $\mu = 3$ and $\lambda = -1$. Equation (4) is satisfied for these values, i.e. $-1 = -1$.

$\Rightarrow$ The two lines intersect in

$$\boxed{P(-3, 0, -1) = \mathcal{L}_1 \cap \mathcal{L}_2}.$$

$\boxed{6}$

(ii) $\mathcal{L}_1$ is parallel to the vector $\vec{v}_1 = 2\vec{\imath} + \vec{\jmath} + 3\vec{k}$
$\mathcal{L}_2$ is parallel to the vector $\vec{v}_2 = -\vec{\imath} + 3\vec{\jmath} + \vec{k}$
$\Rightarrow \vec{v}_1 \times \vec{v}_2$ is perpendicular to $\mathcal{L}_1$ and $\mathcal{L}_2$
We compute

$$\vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 2 & 1 & 3 \\ -1 & 3 & 1 \end{vmatrix} = -8\vec{\imath} - 5\vec{\jmath} + 7\vec{k}$$

$\Rightarrow \mathcal{P}_1 : \; -8x - 5y + 7z = d$ for some $d \in \mathbb{R}$
We have $P \in \mathcal{P}_1$ for say $\lambda = 0$ in (1) $P(-1,1,2) \Rightarrow 8 - 5 + 14 = d \Rightarrow d = 17$.
$\Rightarrow$

$$\boxed{\mathcal{P}_1 : \; -8x - 5y + 7z = 17}.$$

$\boxed{6}$

(iii) Taking $P(x, y, z)$ to be an arbitrary point in the plane $\mathcal{P}_2$, the following vectors are in this plane:

$$\begin{aligned} \overrightarrow{AB} &= 2\vec{\imath} + \vec{\jmath} - \vec{k} \in \mathcal{P}_2 \\ \overrightarrow{AC} &= 3\vec{\imath} + 2\vec{\jmath} + 4\vec{k} \in \mathcal{P}_2 \\ \overrightarrow{AP} &= x\vec{\imath} + (y-3)\vec{\jmath} + (z-1)\vec{k} \in \mathcal{P}_2 \end{aligned}$$

The vector $\overrightarrow{AB} \times \overrightarrow{AC}$ is perpendicular to the plane, such that $\overrightarrow{AP} \cdot \overrightarrow{AB} \times \overrightarrow{AC} = 0$.
Compute

$$\overrightarrow{AP} \cdot \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} x & y-3 & z-1 \\ 2 & 1 & -1 \\ 3 & 2 & 4 \end{vmatrix} = 0$$

$\Rightarrow$The plane containing the points A, B, C is

$$\boxed{\mathcal{P}_2 : \; 32 + 6x - 11y + z = 0}.$$

$\boxed{8}$

(iv) A normal vector to $\mathcal{P}_1$ is $\vec{\eta}_1 = -8\vec{\imath} - 5\vec{\jmath} + 7\vec{k}$.
A normal vector to $\mathcal{P}_2$ is $\vec{\eta}_2 = 6\vec{\imath} - 11\vec{\jmath} + \vec{k}$.
$\Rightarrow \vec{\eta}_1 \times \vec{\eta}_2$ is parallel to $\mathcal{L} = \mathcal{P}_1 \cap \mathcal{P}_2$.
Compute

$$\vec{\eta}_1 \times \vec{\eta}_2 = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ -8 & -5 & 7 \\ 6 & -11 & 1 \end{vmatrix} = 72\vec{\imath} + 50\vec{\jmath} + 118\vec{k}$$

Any point on the line has to satisfy the two equations

$$\begin{aligned} 6x - 11y + z &= -32 \\ -8x - 5y + 7z &= 17. \end{aligned}$$

Taking $y = 0$ gives as solution $x = 241/50$ and $z = -77/25$.
$\Rightarrow$ The line of intersection is

$$\boxed{\mathcal{L}: \quad \frac{x-241/62}{72} = \frac{y}{50} = \frac{z+77/25}{118}} \quad .$$

Equivalently, taking $z = 0$ gives as solution $x = -347/118$ and $z = -77/59$.
$\Rightarrow$ The line of intersection is

$$\boxed{\mathcal{L}: \quad \frac{x+347/118}{72} = \frac{y-77/59}{50} = \frac{z}{118}} \quad .$$

Equivalently, taking $x = 0$ gives as solution $x = -347/118$ and $z = -77/59$.
$\Rightarrow$ The line of intersection is

$$\boxed{\mathcal{L}: \quad \frac{x}{72} = \frac{y-241/72}{50} = \frac{z-347/72}{118}} \quad .$$

$\boxed{6}$