Charles Young '05.

Non-local charges, Graded Lie algebras and Coset-space actions.

hep-th/0503008

- $AdS_5 \times S^5$
  - $\leadsto$ • Bosonic σ-model
  - • Superstring MT hepth/98050

- Classical Integrability BPR hep-th/0305116
- $\mathbb{Z}_3$ models
- $\mathbb{Z}_m$ models

$$AdS_5 \times S^5 = \frac{SO(4,2)\times SO(6)}{SO(4,1)\times SO(5)} = \frac{SU(2,2)\times SU(4)}{SO(4,1)\times SO(5)}$$

$PSU(2,2|4)$

σ-model on coset space G/H

$G = \cancel{SO(4,2)\times SO(6)}$

$H = SO(4,1)\times SO(5)$

$\mathbb{Z}_4$ Grading: $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q} \oplus \mathfrak{m} \oplus \bar{\mathfrak{q}}$

Field: $g(x,t) \in G$

global symmetry: $g \to Ug \quad U \in G$

$$j = g^{-1}dg = A + q + k + \bar{q}$$

local symmetry: $g \to g\,h(x,t) \quad h \in H.$

$$\Rightarrow \quad A \to h^{-1}Ah + h^{-1}dh$$
$$k \to h^{-1}kh.$$
$$q \to h^{-1}qh$$
$$\bar{q} \to h^{-1}\bar{q}h$$

Action

$$S = \int d^2x \, \tfrac{1}{2}\mathrm{tr}\, k_\mu k^\mu \neq \int d^2x \, \mathrm{tr}\,(j-A)_\mu (j-A)^\mu$$

IIB superstring σ-model

WZ term

Green Schwarz: $\int d^2x \, \frac{1}{2} \mathrm{str}\left(k_\mu k^\mu + \boxed{\gamma \varepsilon^{\mu\nu} q_\mu \bar{q}_\nu}\right)$

"Hybrid": $\int d^2x \, \frac{1}{2} \mathrm{str}\left(k_\mu k^\mu + 2 q_\mu \bar{q}^\mu + \gamma \varepsilon^{\mu\nu} q_\mu \bar{q}_\nu\right)$

G-S is kappa-symmetric at $\gamma = \pm 1$

$$\delta g = g\left([K_\mu, k^\mu] + [\bar{K}_\mu, k^\mu] \right)$$

$$K_\mu = +\varepsilon_{\mu\nu} K^\nu, \quad K_\mu \in \mathfrak{q}$$
$$\bar{K}_\mu = -\varepsilon_{\mu\nu} \bar{K}^\nu, \quad \bar{K}_\mu \in \bar{\mathfrak{q}}.$$

Hybrid action aquires $N=2$ superconformal symmetry at $\gamma = \pm 1$

(see BBHZZ /9907200)

Integrability + Flat Currents.

$$j = g^{-1}dg \quad \Rightarrow \quad dj + j \wedge j \equiv 0.$$

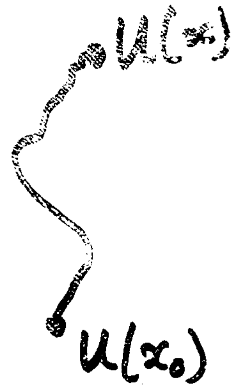
Consider deformation

$$j(\mu) = A + a(\mu)\,k + b(\mu)\,q + c(\mu)\,\bar{q}$$
$$+ d(\mu) * k + e(\mu) * q + f(\mu) * \bar{q}$$

Find $\exists$ 1-parameter family of flat j

(Green Schwarz BPR action
hybrid/pure spinor Vallilo 0307018)

$$dj(\mu) + j(\mu) \wedge j(\mu) = 0$$



$\Rightarrow \quad D^{(\mu)} = d + j(\mu)$ is flat

$\Rightarrow$ equation $D^{(\mu)}u = 0$ is integrable

Monodromy matrix $\quad T^{(\mu)}(t) = P\exp \int_{(-\infty,t)}^{(+\infty,t)} j_1^{(\mu)}(x,t)\,dx$

conserved.

$\mathbb{Z}_3$ - models

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q} \oplus \bar{\mathfrak{q}}$$

$$j = A + q + \bar{q}$$

eg $SL(3,\mathbb{R})/U(1)^2$

$$\begin{pmatrix} p & & \\ & q & \\ & & -p-q \end{pmatrix} \in \mathfrak{h} \qquad \begin{pmatrix} & & c \\ a & & \\ & b & \end{pmatrix} \in \bar{\mathfrak{q}}$$

$$\begin{pmatrix} & d & \\ & & e \\ f & & \end{pmatrix} \in \mathfrak{q}$$

Action:

$$S = \int \mathrm{tr}\Big(\underbrace{q \wedge * \bar{q}}_{= \frac{1}{2}(j-A)\wedge *(j-A)} + \frac{\gamma}{3} \underbrace{q \wedge \bar{q}}_{\text{unique WZ term.}} \Big)$$

Eqns. of Motion:

$$D * q - \gamma D q = 0$$
$$D * \bar{q} + \gamma D \bar{q} = 0$$

$$\left(\text{when } \gamma = 1, \quad D_- q_+ = D_+ \bar{q}_- = 0 \right)$$

Zero curvature condition:

$$dj + j \wedge j = 0 \quad \Rightarrow \quad \begin{aligned} & F + q \wedge \bar{q} + \bar{q} \wedge q = 0 \\ & D q + \bar{q} \wedge \bar{q} = 0 \\ & D \bar{q} + q \wedge q = 0 \end{aligned}$$

Flat Currents:

Let $$j(\mu) = A + e(\mu)\, q + \bar{e}(\mu)\, \bar{q}$$
$$+ f(\mu) * q \quad \bar{f}(\mu) * \bar{q}$$

Require

$$0 = dj(\mu) + j(\mu) \wedge j(\mu)$$
$$= F + (e\bar{e} - f\bar{f})(q \wedge \bar{q} + \bar{q} \wedge q)$$
$$+ (e\bar{f} - \bar{e} f)(q \wedge * \bar{q} + \bar{q} \wedge * q) \Big] \; 0$$
$$+ e Dq + f D * q + (\bar{e}^2 - \bar{f}^2)\, \bar{q} \wedge \bar{q} \Big] \; 1$$
$$+ \bar{e} D\bar{q} + \bar{f} D * \bar{q} + (e^2 - f^2)\, q \wedge q \Big] \; 2$$

$$\Rightarrow \quad (e+f)(\bar{e} - \bar{f}) = (e-f)(\bar{e} + \bar{f}) = 1,$$
$$(e-f)(e+f) = \bar{e} - \bar{f}, \; (\bar{e} - \bar{f})(\bar{e} + \bar{f}) = e + f.$$

$$e + f = \mu \qquad \bar{e} + \bar{f} = \frac{1}{\mu^2}$$

$$e - f = \mu^2 \qquad \bar{e} - \bar{f} = \frac{1}{\mu}$$

$\mathbb{Z}_m$ model $\quad m = 2n+1$

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}_1 \oplus \cdots \oplus \mathfrak{q}_{m-1}$$

$$j = A + q_{(1)} + \cdots + q_{(m-1)}$$

Action: $$S = \int \sum_1^n tr\left(\beta_i\, q_{(i)} \wedge * q_{(m-i)} + \gamma_i\, q_{(i)} \wedge q_{(m-}\right.$$

$\hookrightarrow \beta_i = 1$ gives σ-model kinetic term

$tr(j-A) \wedge * (j-A)$

E.O.M.:

$$D * q_{(1)} =$$

$$\vdots$$

$$D * q_{(m-1)} =$$

Zero curvature cond.:

$$dj + j\wedge j = 0 \quad \Rightarrow \quad F + \sum_1^{m-1} q_{(i)} \wedge q_{(m-i)} = 0$$

$$Dq_{(k)} + \sum_{\substack{i+j \equiv k \\ \{i,j\} \subset \{1,2,\ldots,m-1\}}} q_{(i)} \wedge q_{(j)} = 0$$

Try

$$j(\mu) = A + \sum_{i=1}^{m-1} e_i(\mu)\, q_{(i)} + f_i(\mu) * q_{(i)}$$

Find: $\forall k = 1, 2, \ldots, m-1$

$$(e_k + f_k)(e_{m-k} - f_{m-k}) = 1$$

$$\forall i+j=k \quad (e_i + f_i)(e_j - f_j) = e_k + C^{ij}_k f_k$$

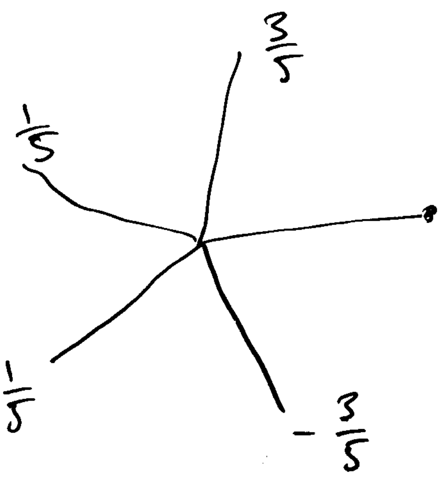
Try: $\boxed{e_k + f_k = \mu^k \qquad e_k - f_k = \frac{1}{\mu^{m-k}}}$

eg $\mathbb{Z}_5$ case

<u>Require</u>: $C^{ij}_k = \begin{cases} +1 & i+j > m \\ -1 & i+j < m \end{cases}$

is possible, with $\gamma_k = 1 - \frac{2k}{m}$

eg $\mathbb{Z}_5$:



Conclusion

For correct WZ term,
σ-model on any (super) coset space
G/H defined by a $\mathbb{Z}_m$ grading
of $\mathfrak{g}$ is classically integrable.

- examples of interest?
- quantum level?