CORRELATION FUNCTIONS FOR INTEGRABLE SPIN CHAINS.

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1. Density matrix for inhomogeneous XXX-model.

Consider the XXX anti-ferromagnet given by the Hamiltonian

$$H_{XXX} = \frac{1}{2}\sum_j \left(\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y + \sigma_j^z\sigma_{j+1}^z\right).$$

The density matrix for finite sub-chain:

$$d_n = \frac{1}{2^n}\sum_{\alpha_1,\cdots,\alpha_n=0}^{3} \langle\mathrm{vac}|\sigma_1^{\alpha_1}\cdots\sigma_n^{\alpha_n}|\mathrm{vac}\rangle \left(\sigma^{\alpha_1}\otimes\cdots\otimes\sigma^{\alpha_n}\right)^T$$

For $1\le k<l\le n$

$$\langle\mathrm{vac}|\sigma_k^{\alpha}\ \sigma_l^{\beta}|\mathrm{vac}\rangle = \mathrm{tr}\left(\sigma_k^{\alpha}\ \sigma_l^{\beta}\ d_n\right)$$

Introduce

$$(h_n)_{\epsilon_1,\cdots,\epsilon_n\bar\epsilon_n,\cdots\bar\epsilon_1} = \prod c_{\bar\epsilon_j,\epsilon'_j}\,(d_n)_{\epsilon_1,\cdots,\epsilon_n}^{\epsilon'_1,\cdots,\epsilon'_n}$$

where

$$c = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

The main result:

$$h_n = e^{\widehat\Omega_n}\,\mathbf{s}_n, \quad \left(\mathbf{s}_n = s_{1,\bar 1}\cdots s_{n,\bar n}\right)$$

The main property of $\widehat\Omega_n$:

$$\widehat\Omega_n^{\left[\frac{n}{2}\right]+1} = 0$$

R-matrix:

$$R(\lambda) = \rho(\lambda)\frac{r(\lambda)}{\lambda+1},$$

where

$$\rho(\lambda) = -\frac{\Gamma\left(\frac{\lambda}{2}\right)\Gamma\left(-\frac{\lambda}{2}+\frac{1}{2}\right)}{\Gamma\left(-\frac{\lambda}{2}\right)\Gamma\left(\frac{\lambda}{2}+\frac{1}{2}\right)}$$

Transfer matrix

$$t_N(\lambda) = \mathrm{tr}\left(R_{\alpha,-N+1}(\lambda)\cdots R_{\alpha,N}(\lambda)\right),$$

Hamiltonians:

$$\log t_N(0)^{-1}t_N(\lambda) = \sum_{k=1}^{\infty} I_k\lambda^k,$$

Inhomogeneous transfer matrix

$$\begin{aligned} t(\lambda|\lambda_1,\cdots,\lambda_n) = \mathrm{tr}\Big(&R_{a,-N+1}(\lambda)\cdots R_{a,0}(\lambda) \\ &\times R_{a,1}(\lambda-\lambda_1)\cdots R_{a,n}(\lambda-\lambda_n) \\ &\times R_{a,n+1}(\lambda)\cdots R_{a,N}(\lambda)\Big), \end{aligned}$$

Thermodynamical limit: $N \to \infty$. Main result:

$$h_n(\lambda_1,\cdots,\lambda_n) = \exp\Big(\sum_{i<j}\widehat{\Omega}_n^{(i,j)}(\lambda_1,\cdots,\lambda_n)\Big)$$

The main properties:

$$\left[\widehat{\Omega}_n^{(i,j)}(\lambda_1,\cdots,\lambda_n)\ ,\ \widehat{\Omega}_n^{(k,l)}(\lambda_1,\cdots,\lambda_n)\right] = 0$$

$$\widehat{\Omega}_n^{(i,j)}(t_1,\cdots,t_n)\widehat{\Omega}_n^{(k,l)}(t_1,\cdots,t_n) = 0,\ \{i,j\}\cap\{k,l\} \neq \emptyset$$

2. Construction of $\widehat{\Omega}^{(i,j)}$.

Explicitly,

$$\widehat{\Omega}_n^{(i,j)}(\lambda_1, \cdots, \lambda_n) = \omega(\lambda_{ij})\widehat{X}_n^{(i,j)}(\lambda_1, \cdots, \lambda_n)$$

where

$$\omega(\lambda) = 4\frac{\partial}{\partial \lambda} \log \rho(\lambda) + \frac{2}{(\lambda^2 - 1)} =$$

$$= \sum_{k=1}^{\infty} (-1)^k \frac{8k}{\lambda^2 - k^2} + \frac{2}{(\lambda^2 - 1)}$$

$\widehat{X}_n^{(i,j)}(\lambda_1, \cdots, \lambda_n)$ is rational function of (λ_k) with simple poles at $\lambda_i = \lambda_k$, $\lambda_j = \lambda_k$, $k \neq i, j$ only. In homogeneous limit $\lambda_i \to 0$ one gets:

$$\Omega_n = \lim_{\lambda_i \to 0} \sum_{i<j} \widehat{\Omega}_n^{(i,j)}(\lambda_1, \cdots, \lambda_n) = \sum_{k=0}^{n-1} Q_k \, \zeta_a(2k+1)$$

where

$$\zeta_a(s) = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{k^s} = \left(1 - 2^{-s}\right) \zeta(s)$$

The reason for that is

$$\omega(\lambda) = \frac{2}{(\lambda^2 - 1)} - 8 \sum_{m=0}^{\infty} \lambda^{2m} \zeta_a(2m+1)$$

3. Definition of $\widehat{X}_n^{(i,j)}(\lambda_1,\cdots,\lambda_n)$.

1. Trace.

We define "trace over a space of fractional dimension": unique $\mathbb{C}[x]$ linear map

$$\mathrm{Tr}_x \;:\; U(\mathfrak{sl}_2)\otimes\mathbb{C}[x]\longrightarrow\mathbb{C}[x]$$

such that for any non-negative integer k we have

$$\mathrm{Tr}_{k+1}(A)=\mathrm{tr}_{V^{(k)}}\pi^{(k)}(A)\qquad(A\in U(\mathfrak{sl}_2))$$
$$\mathrm{Tr}_x(AB)=\mathrm{Tr}_x(BA),\quad \mathrm{Tr}_x(1)=x,$$
$$\mathrm{Tr}_x(A)=0\quad \text{if } A \text{ has non-zero weight},$$
$$\mathrm{Tr}_x(e^{zH})=\frac{\sinh(xz)}{\sinh z},$$
$$\mathrm{Tr}_x\left(\Big(\frac{H^2}{2}+H+2FE\Big)A\right)=\frac{x^2-1}{2}\mathrm{Tr}_x(A)$$

The point is that from

$$[E,F]=H,\quad 2FE+H+\frac{H^2}{2}=C$$

one finds:

$$FE=\frac{C}{2}-\frac{H}{2}-\frac{H^2}{4},\quad EF=\frac{C}{2}+\frac{H}{2}-\frac{H^2}{4}$$

Properties:

$$\mathrm{Tr}_x(A)=\mathrm{Tr}_{-x}(A),$$

$$\mathrm{Tr}_x(A)-x\varepsilon(x)\in x(x^2-1)\mathbb{C}[x]$$

L-operator:

$$L(\lambda) = \begin{pmatrix} \lambda + \frac{1+H}{2} & F \\ E & \lambda + \frac{1-H}{2} \end{pmatrix} \in U(\mathfrak{sl}_2) \otimes \mathbb{C}^2$$

Consider a 'transfer matrix'

$$\widehat{X}_n(\lambda_1, \cdots, \lambda_n) = \frac{1}{\lambda_{1,2} \prod_{p=3}^n \lambda_{1,p} \lambda_{2,p}}$$
$$\times \mathrm{Tr}_{\lambda_{1,2}} \Big(T_n^{[1]} \big(\frac{\lambda_1 + \lambda_2}{2}; \lambda_1, \cdots, \lambda_n \big) \Big) P_{12} \mathcal{P}^-_{1\bar{1}} \mathcal{P}^-_{2\bar{2}}.$$

where

$$T_n^{[1]}(\lambda; \lambda_1, \ldots, \lambda_n)$$
$$= L_{\bar{2}}(\lambda - \lambda_2 - 1) \cdots L_{\bar{n}}(\lambda - \lambda_n - 1)$$
$$\times L_n(\lambda - \lambda_n) \cdots L_2(\lambda - \lambda_2).$$

For $i < j$, we define

$$\widehat{X}_n^{(i,j)}(\lambda_1, \cdots, \lambda_n) = \widehat{X}_n^{(j,i)}(\lambda_1, \cdots, \lambda_n) =$$
$$= \mathbb{R}_n^{(i,j)}(\lambda_1, \cdots, \lambda_n) \widehat{X}_n(\lambda_i, \lambda_j, \lambda_1, \cdots, \widehat{\lambda_i}, \cdots, \widehat{\lambda_j}, \cdots, \lambda_n)$$
$$\times \mathbb{R}_n^{(i,j)}(\lambda_1, \cdots, \lambda_n)^{-1}.$$

Here $\mathbb{R}_n^{(i,j)}$ is defined using $\check{R} = PR$:

$$\mathbb{R}_n^{(i,j)}(\lambda_1, \cdots, \lambda_n)$$
$$= \check{R}_{i,i-1}(\lambda_{i,i-1}) \cdots \check{R}_{2,1}(\lambda_{i,1})$$
$$\times \check{R}_{j,j-1}(\lambda_{j,j-1}) \cdots \widehat{\check{R}_{i+1,i}(\lambda_{j,i})} \cdots \check{R}_{3,2}(\lambda_{j,1})$$
$$\times \check{R}_{\overline{i-1},\bar{i}}(\lambda_{i-1,i}) \cdots \check{R}_{\bar{1}\bar{2}}(\lambda_{1,i})$$
$$\times \check{R}_{\overline{j-1},\bar{j}}(\lambda_{j-1,j}) \cdots \widehat{\check{R}_{\bar{i},\overline{i+1}}(\lambda_{i,j})} \cdots \check{R}_{\bar{2},\bar{3}}(\lambda_{1,j}).$$

4. Properties of $\widehat{X}_n^{(i,j)}(\lambda_1, \cdots, \lambda_n)$.

1. Exchange relation.

$$\check{R}_{k,k+1}(\lambda_{k,k+1})\check{R}_{\bar{k+1},\bar{k}}(\lambda_{k+1,k})\widehat{X}_n^{(i,j)}(\cdots, \lambda_k, \lambda_{k+1}, \cdots) =$$
$$= \widehat{X}_n^{(\pi_k(i),\pi_k(j))}(\cdots, \lambda_{k+1}, \lambda_k, \cdots)\check{R}_{k,k+1}(\lambda_{k,k+1})\check{R}_{\bar{k+1},\bar{k}}(\lambda_{k+1,k})$$

2. Two difference equations.

$$\widehat{X}_n^{(1,j)}(\lambda_1 - 1, \cdots, \lambda_n) =$$
$$= A_n(\lambda_1, \cdots, \lambda_n)\widehat{X}_n^{(1,j)}(\lambda_1, \cdots, \lambda_n,)$$

$$\widehat{X}_n^{(i,j)}(\lambda_1 - 1, \cdots, \lambda_n) =$$
$$= A_n(\lambda_1, \cdots, \lambda_n)\widehat{X}_n^{(1,j)}(\lambda_1, \cdots, \lambda_n)A_n(\lambda_1, \cdots, \lambda_n)^{-1}$$

where

$$A_n(\lambda_1, \cdots, \lambda_n) = (-)^n R_{\bar{1}\bar{2}}(\lambda_{12} - 1)\cdots R_{\bar{1}\bar{n}}(\lambda_{1n} - 1)P_{1\bar{1}}$$
$$\times R_{1n}(\lambda_{1n})\cdots R_{12}(\lambda_{12})$$

3. Commutation relation.

$$\left[\widehat{X}_n^{(i,j)}(\lambda_1, \cdots, \lambda_n), \widehat{X}_n^{(k,l)}(\lambda_1, \cdots, \lambda_n)\right] = 0$$

4. Nilpotency.

$$\widehat{X}_n^{(i,j)}(\lambda_1, \cdots, \lambda_n)\widehat{X}_n^{(k,l)}(\lambda_1, \cdots, \lambda_n) = 0$$
$$\text{i}f \quad \{i,j\} \cap \{k,l\} \neq \emptyset$$

5. Recurrence Relation.

$$\mathcal{P}^{-}\widehat{X}_n^{(1,j)}(\lambda_1,\cdots,\lambda_n)=0,$$

$$\mathcal{P}^{-}\widehat{X}_n^{(i,j)}(\lambda_1,\lambda_2,\cdots,\lambda_n)=\widehat{X}_{n-1}^{(i,j)}(\lambda_2,\cdots,\lambda_n)\mathcal{P}^{-},\quad i,j\neq 1$$

6. Cancellation Identity.

$$\Big(\sum_{j=2}^{n}\widehat{Y}_n^{(1,j)}(\lambda_1,\cdots,\lambda_n)+\big(A_n(\lambda_1,\cdots,\lambda_n)^{-1}-1\big)\Big)\mathbf{s}_n=0$$

where

$$\widehat{Y}_n^{(1,j)}(\lambda_1,\cdots,\lambda_n)=\frac{3}{(\lambda_{1j}^2-1)\lambda_{1j}(\lambda_{1j}-2)}\widehat{X}_n^{(i,j)}(\lambda_1,\cdots,\lambda_n)$$

The meaning of $\widehat{Y}_n^{(1,j)}$ is the following. Recall that

$$\widehat{\Omega}_n^{(1,j)}(\lambda_1,\cdots,\lambda_n)=\omega(\lambda_{1j})\widehat{X}_n^{(1,j)}(\lambda_1,\cdots,\lambda_n)$$

$\omega(\lambda)$ satisfies simple difference equation:

$$\omega(\lambda-1)+\omega(\lambda)=\frac{3}{(\lambda^2-1)\lambda(\lambda-2)}$$

Hence

$$\widehat{\Omega}_n^{(1,j)}(\lambda_1-1,\cdots,\lambda_n)=A_n(\lambda_1,\cdots,\lambda_n)$$
$$\times\Big(\widehat{\Omega}_n^{(1,j)}(\lambda_1,\cdots,\lambda_n,)+\widehat{Y}_n^{(1,j)}(\lambda_1,\cdots,\lambda_n,)\Big)$$

5. Equations for $h_n(\lambda_1, \cdots, \lambda_n)$.

According to Kyoto school $h_n(\lambda_1, \cdots, \lambda_n)$ satisfies the following requirements:

1. Invariance. $h_n(\lambda_1, \cdots, \lambda_n)$ is invariant under the action of $\mathfrak{sl}_2$

2. Symmetry.

$$h_n(\cdots, \lambda_{j+1}, \lambda_j, \cdots)$$
$$= \check{R}_{j,j+1}(\lambda_{j,j+1})\check{R}_{\overline{j+1},\bar{j}}(\lambda_{j+1,j})h_n(\cdots, \lambda_j, \lambda_{j+1}, \cdots)$$

3. Difference equation.

$$h_n(\lambda_1 - 1, \lambda_2, \cdots, \lambda_n)$$
$$= A_{\bar{1}}(\lambda_1, \cdots, \lambda_n)h_n(\lambda_1, \lambda_2, \cdots, \lambda_n)$$

4. Normalization condition.

$$\mathcal{P}^{-}_{1,\bar{1}} \cdot h_n(\lambda_1, \cdots, \lambda_n) =$$
$$= (-1)^{n-1} s_{1,\bar{1}}\, h_{n-1}(\lambda_2, \cdots, \lambda_n)$$

Why our Ansatz satisfy these requirement?
Invariance is obvious.
Symmetry follows from **Exchange relation** and

$$\check{R}_{j,j+1}(\lambda_{j,j+1})\check{R}_{\overline{j+1},\bar{j}}(\lambda_{j+1,j})\mathbf{s}_n = \mathbf{s}_n.$$

Difference equation. Rewrite the Ansatz as

$$\begin{aligned} &h_n(\lambda_1,\cdots,\lambda_n) = \\ &= \{\sum_{k=0}^{\infty} \sum_{\substack{i_k>\cdots>i_1 \\ j_p>i_p}} \widehat{\Omega}^{(i_k,j_k)}(\lambda_1,\cdots,\lambda_n)\cdots\widehat{\Omega}^{(i_1,j_1)}(\lambda_1,\cdots,\lambda_n) = \\ &= \sum_{k=0}^{\infty} \sum_{\substack{i_k>\cdots>i_1 \\ j_p>i_p, i_1\neq 1}} \widehat{\Omega}^{(i_k,j_k)}(\lambda_1,\cdots,\lambda_n)\cdots\widehat{\Omega}^{(i_1,j_1)}(\lambda_1,\cdots,\lambda_n) + \\ &+ \sum_{k=0}^{\infty} \sum_{\substack{i_k>\cdots>i_2 \\ j_p>i_p}} \widehat{\Omega}^{(i_k,j_k)}(\lambda_1,\cdots,\lambda_n)\cdots\widehat{\Omega}^{(1,j)}(\lambda_1,\cdots,\lambda_n)\}\mathbf{s}_n \end{aligned}$$

Hence

$$h_n(\lambda_1-1,\cdots,\lambda_n)=A_n(\lambda_1,\cdots,\lambda_n)$$

$$\times\{\sum_{k=0}^{\infty}\sum_{\substack{i_k>\cdots>i_1\\ j_p>i_p,i_1\neq 1}}\widehat{\Omega}^{(i_k,j_k)}(\lambda_1,\cdots,\lambda_n)\cdots\widehat{\Omega}^{(i_1,j_1)}(\lambda_1,\cdots,\lambda_n)$$

$$\times A_n(\lambda_1,\cdots,\lambda_n)^{-1}+$$

$$+\sum_{k=0}^{\infty}\sum_{\substack{i_k>\cdots>i_2\\ j_p>i_p}}\widehat{\Omega}^{(i_k,j_k)}(\lambda_1,\cdots,\lambda_n)\cdots\widehat{\Omega}^{(1,j)}(\lambda_1,\cdots,\lambda_n)$$

$$+\sum_{k=0}^{\infty}\sum_{\substack{i_k>\cdots>i_2\\ j_p>i_p}}\widehat{\Omega}^{(i_k,j_k)}(\lambda_1,\cdots,\lambda_n)\cdots\widehat{Y}^{(1,j)}(\lambda_1,\cdots,\lambda_n)\Big\}\mathbf{s}_n$$

Recall the Cancellation identity:

$$\sum_{j\neq 1}\widehat{Y}^{(1,j)}(\lambda_1,\cdots,\lambda_n)\mathbf{s}_n=-A_n(\lambda_1,\cdots,\lambda_n)^{-1}\mathbf{s}_n+\mathbf{s}_n$$

Collecting terms one finds:

$$h_n(\lambda_1-1,\cdots,\lambda_n)=A_n(\lambda_1,\cdots,\lambda_n)h_n(\lambda_1,\cdots,\lambda_n)$$

QED

6. Generalisation to XYZ-case.

Here we use parameter t:

$$\lambda = \eta t$$

The R matrix is given by

$$R(t) = \rho(t)\frac{r(t)}{[t+\eta]}$$

$$r(t) = \frac{1}{2}\sum_{\alpha=0}^{3}\frac{\theta_{\alpha+1}(2t+\eta)}{\theta_{\alpha+1}(\eta)}\sigma^{\alpha}\otimes\sigma^{\alpha},$$

where

$$[t] = \frac{\theta_1(2t)}{\theta_1(2\eta)}.$$

Consider for simplicity only disordered regime:

$$\eta, t \in i\mathbb{R}, \quad -i\eta > 0$$

$\rho(t)$ is given by

$$\rho(t) = e^{-2\pi i t} \times \frac{\gamma(2\eta - 2t)}{\gamma(2\eta + 2t)}\frac{\gamma(4\eta + 2t)}{\gamma(4\eta - 2t)},$$

$$\gamma(u) = \Gamma(u, 4\eta, \tau),$$

where

$$\Gamma(u,\sigma,\tau) = \prod_{j,k=0}^{\infty}\frac{1 - e^{2\pi i((j+1)\tau + (k+1)\sigma - u)}}{1 - e^{2\pi i(j\tau + k\sigma + u)}}$$

The main formula remains the same, but definition of $\widehat{\Omega}_n^{(i,j)}$ changes:

$$\widehat{\Omega}_n^{(i,j)}(t_1, \cdots, t_n) = \sum_{a=1}^{3} \omega_a(t_{ij}) \widehat{X}_{a,n}^{(i,j)}(t_1, \cdots, t_n)$$

where

$$\omega_1(t) = \frac{\partial}{\partial t} \ln \varphi(t), \ \omega_2(t) = \frac{\partial}{\partial \eta} \ln \varphi(t), \ \omega_3(t) = \frac{\partial}{\partial \tau} \ln \varphi(t)$$

with

$$\varphi(t) = \rho(t)^4 \cdot \frac{\theta_1(2\eta - 2t)}{\theta_1(2\eta + 2t)}.$$

7. Definition of $\widehat{X}^{(i,j)}_{a,n}(t_1,\cdots,t_n)$.

L-operator:

$$L(t)=\frac{1}{2}\sum_{a=0}^{3}\frac{\theta_{a+1}(2t+\eta)}{\theta_{a+1}(\eta)}S_a\otimes\sigma^a \quad \in \mathcal{A}\otimes \mathrm{End}(V).$$

where $\mathcal{A}$ is Sklyanin algebra:

$$[S_0,S_a]=iJ_{bc}(S_bS_c+S_cS_b),$$
$$[S_b,S_c]=i(S_0S_a+S_aS_0),$$

where

$$J_{bc}=-\frac{J_b-J_c}{J_a}=\varepsilon_a\frac{\theta_1(\eta)^2\theta_{a+1}(\eta)^2}{\theta_{b+1}(\eta)^2\theta_{c+1}(\eta)^2},$$
$$J_a=\frac{\theta_{a+1}(2\eta)\theta_{a+1}}{\theta_{a+1}(\eta)^2},\quad \varepsilon_\alpha=\sigma^\alpha\left(\sigma^\alpha\right)^T$$

Casimir elements:

$$K_0=\sum_{a=0}^{3}S_a^2,\quad K_2=\sum_{a=1}^{3}J_aS_a^2.$$

'Transfer matrix' with minor changes

$$\widehat{X}^{(1,2)}_n(t_1,\cdots,t_n)=\frac{1}{[t_{1,2}]\prod_{p=3}^{n}[t_{1,p}][t_{2,p}]}$$
$$\times\mathrm{Tr}_{\frac{t_{1,2}}{\eta}}\left(T_n^{[1]}\left(\frac{t_1+t_2}{2};t_1,\cdots,t_n\right)\right)P_{12}\mathcal{P}^-_{1\bar{1}}\mathcal{P}^-_{2\bar{2}}$$

Then

$$c\widehat{X}^{(1,2)}_{1,n}(t_1,\cdots,t_n) = (1 - t_{1,2}\Delta_1)\,\widehat{X}^{(1,2)}_{n}(t_1,\cdots,t_n),$$

$$c\widehat{X}^{(1,2)}_{2,n}(t_1,\cdots,t_n) = -\eta\Delta_1\widehat{X}^{(1,2)}_{n}(t_1,\cdots,t_n),$$

$$c\widehat{X}^{(1,2)}_{3,n}(t_1,\cdots,t_n) = (\Delta_\tau - \tau\Delta_1)\,\widehat{X}^{(1,2)}_{n}(t_1,\cdots,t_n),$$

where $c = 2\theta_1'/\theta_1(2\eta)$ and

$$\Delta_a f(t_1,\cdots) = f(t_1 + a,\cdots) - f(t_1,\cdots)$$

Traces are reduced to the following ones:

$$\mathrm{Tr}_\lambda 1 = \lambda,$$
$$\mathrm{Tr}_\lambda S_a = 2\delta_{a0}\frac{\theta_1(\lambda\eta)}{\theta_1(2\eta)},$$
$$\mathrm{Tr}_\lambda S_a^2 = \frac{2\varepsilon_a\theta_{a+1}(\eta)^2}{\theta_1'\theta_1(2\eta)^3}\left(\frac{\partial}{\partial t} - \lambda\frac{\partial}{\partial\eta}\right)\theta_{a+1}(t+\eta)\theta_{a+1}(t-\eta)$$