A large class of 3D integrable lattice spin models

G. von Gehlen *ICFT05 London, April 22, 2005*

joint work with S. Pakuliak, S. Sergeev

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Many 2-D integrable spin models known: Ising, RSOS, etc.

Yang-Baxter equation $\Longrightarrow$ commuting Transfer Matrices

$$\mathcal{R}_{12}(u)\,\mathcal{R}_{13}(u+v)\,\mathcal{R}_{23}(v) \;=\; \mathcal{R}_{23}(v)\,\mathcal{R}_{13}(u+v)\,\mathcal{R}_{12}(u)\,.$$

Systematic construction by quantum group techniques.

Analogue of Yang-Baxter equation for *3-dim. integrability*:

Tetrahedron equation (TE)

A.B.Zamolodchikov 1981, Bazhanov-Stroganov 1982

$$\mathcal{R}_{123}\;\;\mathcal{R}_{145}\;\;\mathcal{R}_{246}\;\;\mathcal{R}_{356} = \mathcal{R}_{356}\;\;\mathcal{R}_{246}\;\;\mathcal{R}_{145}\;\;\mathcal{R}_{123}.$$

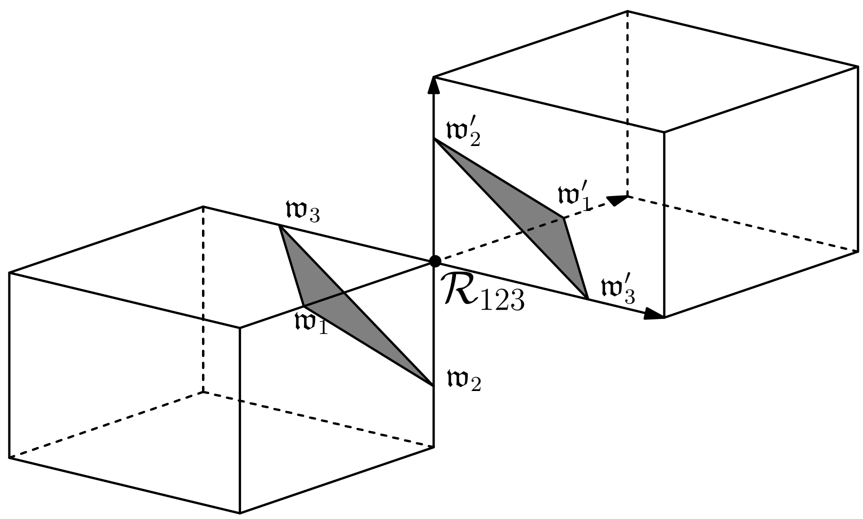


Figure 1: $\mathcal{R}_{123}$ maps initial variables $\mathfrak{w}_1$, $\mathfrak{w}_2$, $\mathfrak{w}_3$ to final $\mathfrak{w}_1'$, $\mathfrak{w}_2'$, $\mathfrak{w}_3'$

More explicitly (Each $\mathcal{R}_{ijk}$ depends on 3 param., there are 5 different param. in the TE):

$$\sum_{j_1\cdots j_6} \mathcal{R}_{i_1,i_2,i_3}^{j_1,j_2,j_3}\,\mathcal{R}_{j_1,i_4,i_5}^{k_1,j_4,j_5}\,\mathcal{R}_{j_2,j_4,i_6}^{k_2,k_4,j_6}\,\mathcal{R}_{j_3,j_5,j_6}^{k_3,k_5,k_6} \sim \sum_{j_1\cdots j_6} \mathcal{R}_{i_3,i_5,i_6}^{j_3,j_5,j_6}\,\mathcal{R}_{i_2,i_4,j_6}^{j_2,j_4,k_6}\,\mathcal{R}_{i_1,j_4,j_5}^{j_1,k_4,k_5}\,\mathcal{R}_{j_1,j_2,j_3}^{k_1,k_2,k_3}$$

$\Longrightarrow$ Layer-transfer matrices commute $\Longrightarrow$ Integrability

TE is very restrictive (N^{12} eqs., by symmetries$\rightarrow$ N^8 eqs.), essentially only one solution known:

$\mathbb{Z}_N$- Zamolodchikov-Baxter-Bazhanov (ZBB) model

for $N>2$ Boltzmann weights complex, models are chiral.

Partition function per site k has been calculated (Baxter 1983)

Turns our to be real:

$$\log\left(k/2\xi\right) \;=\; \frac{1}{2\pi}\sum_{i=1}^{4}\int_0^{\zeta_i}\left[\log(2\cos x)+x\tan x\right]\,dx$$

Strong indication that integrable ZBB model is critical for all 3 parameters, has no temperature variable $\Longrightarrow$ *bad for describing phase transtions.* (Baxter, Forrester 1985)

Need less restrictive "modified" TE-equations (MTE)

first proposed by Mangazeev and Stroganov 1993

different rapidities at the left and right hand sides of TE,

related by *classical integrable equations*.

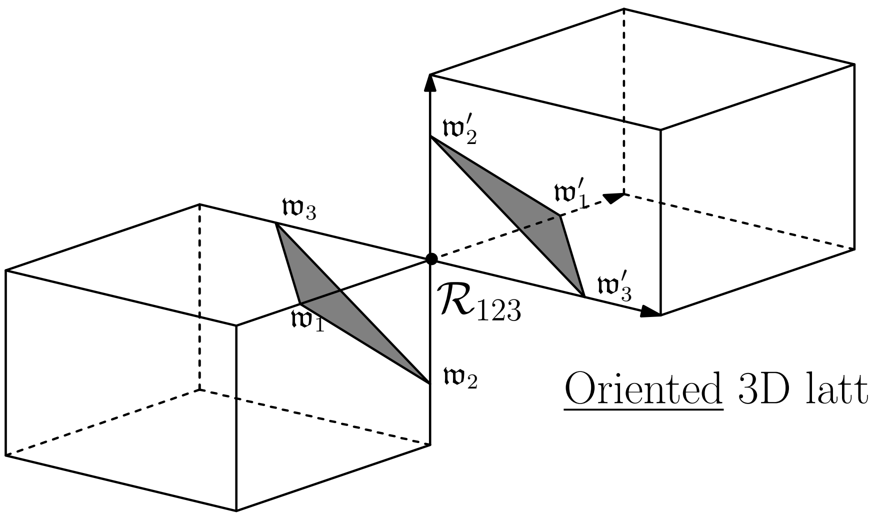
Check of TE usually very tedious.

Sergeev formulation of the 3D vertex ZBB-model:

Quantum variables: elements of ultralocal Weyl algebra at root of unity:

$$\mathbf{u}_j \cdot \mathbf{w}_j = \omega \mathbf{w}_j \cdot \mathbf{u}_j; \quad \omega = e^{2\pi i/N}; \quad \mathbf{u}_i \cdot \mathbf{w}_j = \mathbf{w}_j \cdot \mathbf{u}_i \text{ for } i \neq j.$$

Attach also scalar κ_j to each link, together: $\mathfrak{w}_j = (\mathbf{u}_j, \mathbf{w}_j, \kappa_j)$.



Oriented 3D lattice

Key object: Canonical invertible rational mapping operator $\mathcal{R}_{123}$

$$(\mathcal{R}_{123} \circ \Psi)(\mathbf{u}_1, \mathbf{w}_1, \mathbf{u}_2, \ldots, \mathbf{w}_3) = \Psi(\mathbf{u}'_1, \mathbf{w}'_1, \mathbf{u}'_2, \ldots, \mathbf{w}'_3).$$

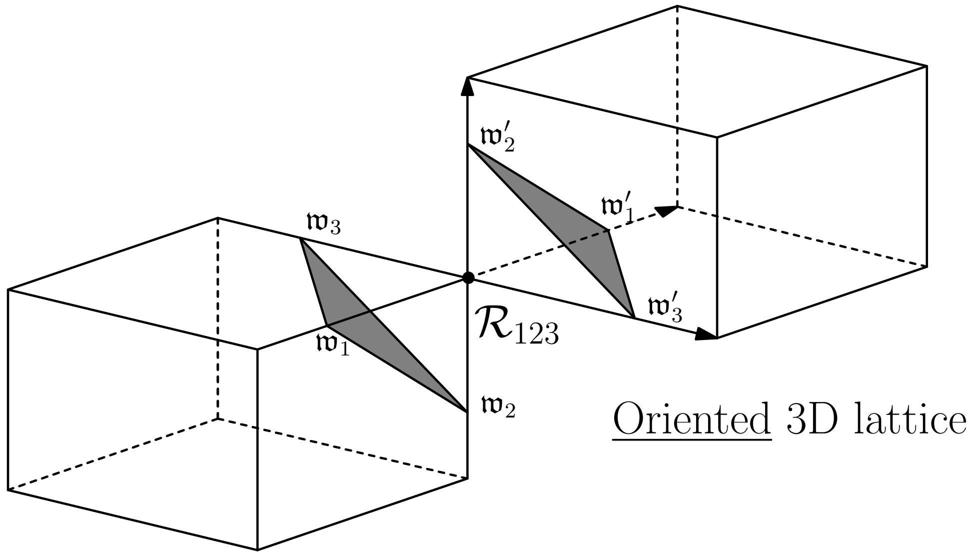
$\mathcal{R}_{123}$ is uniquely determined from postulates:

1: Baxter Z-invariance:

lines may be shifted respect to each other

2: Linear Problem:

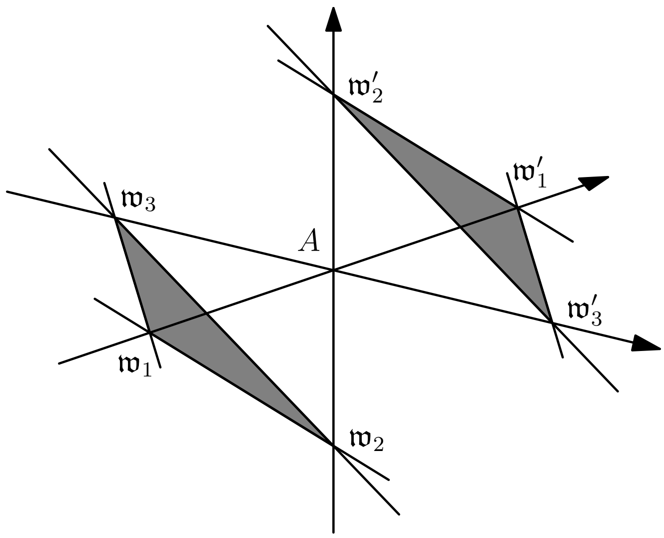
$$0 = \langle \Phi_a | + \omega^{1/2} \langle \Phi_b | \, \mathbf{u}_1 + \langle \Phi_c | \, \mathbf{w}_1 + \kappa_1 \langle \Phi_d | \, \mathbf{u}_1 \, \mathbf{w}_1.$$

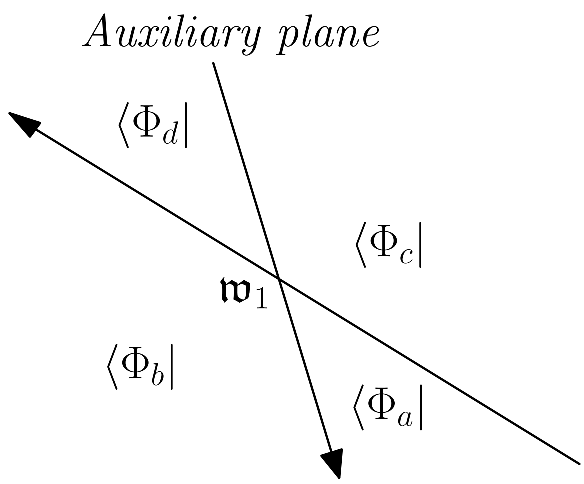


Linear problem considered like Kirchhoff law:

Consider currents $\langle\phi_i|$; flowing out of the vertex $\mathfrak{w}_1$ into the four sectors of the auxiliary plane around them, distributed according to the Weyl variables: (κ_i "coupling constant")

$\omega^{1/2}\langle\phi_i|\,\mathbf{u}_1$ flows *between* the arrows, $\langle\phi_i|\mathbf{w}_1$ *below* arrows, $\langle\phi_i|$ to the *left* sector, $\kappa_i\langle\phi_i|\mathbf{u}_1\mathbf{w}_1$ to the *right* sector



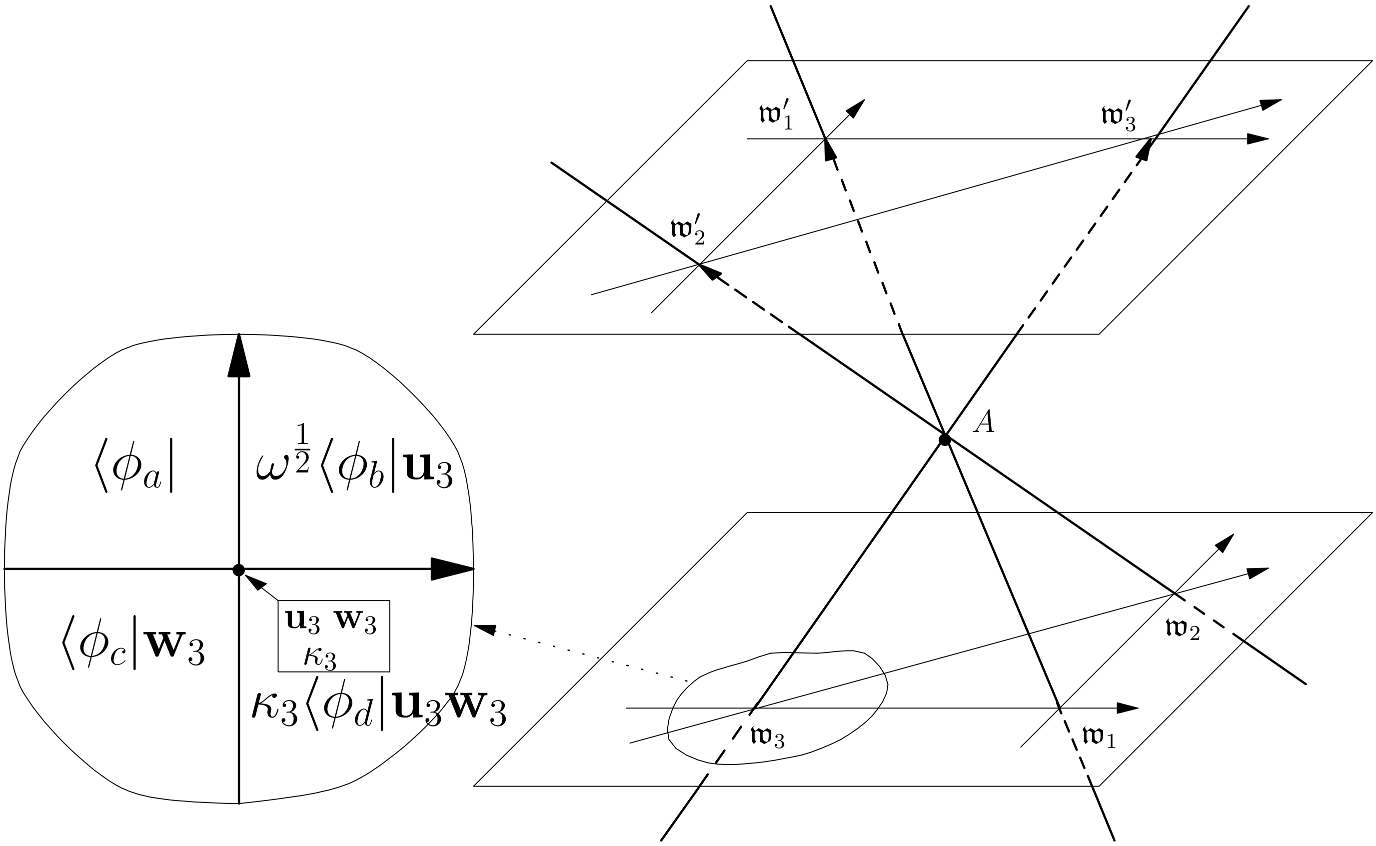


Total current received by an inner sector shall vanish.

Another view of the same auxiliary planes:

A is vertex of the basic lattice

Left: Magnified view of the currents emerging from $\mathfrak{w}_3$

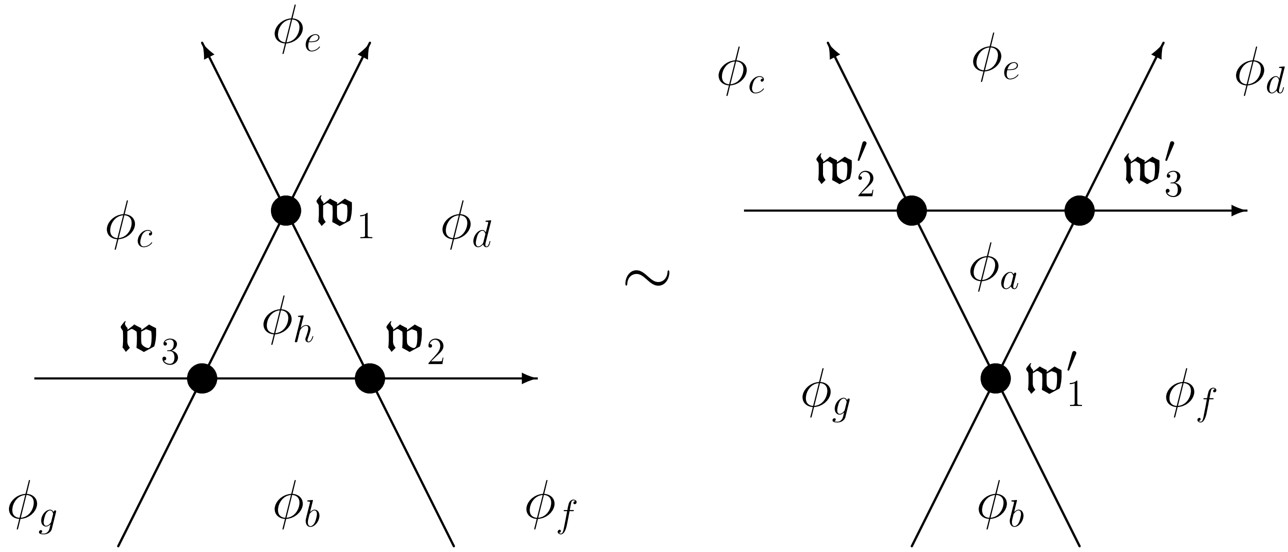


Conditions which determine $\mathcal{R}_{123}$ uniquely as a canonical and invertible map: (S.Sergeev, J.Phys.A32 (1999) 5639)

- *Linear Problem:*

  The total current received by inner sectors shall vanish.

- *Currents in external sectors are invariant against Z-transformation of inner lines.*



Examples from $\Delta = \nabla$:

Total current received by the left internal sector vanishes:

$$\langle\phi_h| \;=\; \mathbf{w}_1\langle\phi_1| \;+\; \langle\phi_2| \;+\; \omega^{1/2}\mathbf{u}_3\langle\phi_3| \;=\; 0\,;$$

Current received by left external sector shall equal that received by corresponding right hand sector:

$$\langle\phi_c| \;=\; \langle\phi'_2| \;=\; \langle\phi_1| \;+\; \langle\phi_3|\,;$$

$$\langle\phi_b| \;=\; \mathbf{w}'_1\langle\phi'_1| \;=\; \mathbf{w}_2\langle\phi_2| \;+\; \kappa_3\mathbf{u}_3\mathbf{w}_3\langle\phi_3|\,;\;\; etc.$$

8 equations: all currents can be eliminated. *Unique solution* to this $\Delta = \nabla$ linear problem: *"Sergeev mapping"*

$$\mathbf{w}'_1 \;=\; \frac{\mathbf{w}_2}{\mathbf{w}_1\mathbf{w}_3^{-1} \;-\; \omega^{1/2}\,\mathbf{u}_3\mathbf{w}_3^{-1} \;+\; \kappa_3\,\mathbf{w}_2^{-1}\mathbf{u}_3}$$

$$\cdots$$

$$\mathbf{u}'_3 \;=\; \frac{\mathbf{u}_2}{\mathbf{u}_1^{-1}\mathbf{u}_3 \;-\; \omega^{1/2}\,\mathbf{u}_1^{-1}\mathbf{w}_1 \;+\; \kappa_1\,\mathbf{w}_1\mathbf{u}_2^{-1}}$$

Complete formulae of the unique solution:

$$\mathbf{w}_1' \;=\; \frac{\mathbf{w}_2}{\mathbf{w}_1\mathbf{w}_3^{-1} \;-\; \omega^{1/2}\,\mathbf{u}_3\mathbf{w}_3^{-1} \;+\; \kappa_3\;\mathbf{w}_2^{-1}\mathbf{u}_3}$$

$$\mathbf{u}_1' \;=\; \frac{\kappa_2\mathbf{w}_3^{-1}}{\kappa_1\,\mathbf{u}_2^{-1}\mathbf{w}_3^{-1} \;+\; \kappa_3\,\mathbf{u}_1^{-1}\mathbf{w}_2^{-1} \;-\; \kappa_1\,\kappa_3\,\omega^{-1/2}\,\mathbf{u}_2^{-1}\mathbf{w}_2^{-1}}$$

$$\mathbf{u}_3' \;=\; \frac{\mathbf{u}_2}{\mathbf{u}_1^{-1}\mathbf{u}_3 \;-\; \omega^{1/2}\,\mathbf{u}_1^{-1}\mathbf{w}_1 \;+\; \kappa_1\;\mathbf{w}_1\mathbf{u}_2^{-1}}$$

$$\mathbf{w}_1'\mathbf{w}_2' \;=\; \mathbf{w}_1\mathbf{w}_2; \qquad \mathbf{u}_2'\mathbf{u}_3' \;=\; \mathbf{u}_2\mathbf{u}_3; \qquad \mathbf{u}_1'\mathbf{w}_3'^{\,-1} \;=\; \mathbf{u}_1\mathbf{w}_3^{-1}$$

These eqs. define the **mapping** $\mathcal{R}_{123}$ (invertible, canonical):

$$(\mathcal{R}_{123}\circ\Phi)\,(\mathbf{u}_1,\mathbf{w}_1,\mathbf{u}_2,\ldots,\mathbf{w}_3) \stackrel{def}{=} \Phi(\mathbf{u}_1',\mathbf{w}_1',\mathbf{u}_2',\ldots,\mathbf{w}_3').$$

for any rational function Φ of the $\mathbf{u}_1,\ldots,\mathbf{w}_3$.

Canonical: maps the triple Weyl algebra of $\mathbf{u}_1,\mathbf{w}_1,\mathbf{u}_2,\ldots,\mathbf{w}_3$

into the same Weyl algebra of the $\mathbf{u}_1',\mathbf{w}_1',\mathbf{u}_2',\ldots,\mathbf{w}_3'$

Functional map $\mathcal{R}_{123}^{(f)}$ implied by $\mathcal{R}_{123}$:

For $\omega = e^{2\pi i/N}$ we have $(a\mathbf{u}+b\mathbf{w})^N = (au)^N + (bw)^N$, so:

$$w_1'^{\,N} \;=\; \frac{w_2^N}{w_1^N w_3^{-N} \;+\; u_3^N w_3^{-N} \;+\; \kappa_3^N w_2^{-N} u_3^N};\quad etc.$$

For any rational *scalar* function $\psi(\ldots)$ we define

$$\left(\mathcal{R}_{123}^{(f)}\circ\psi\right)(u_1,w_1,u_2,\ldots,w_3) \stackrel{def}{=} \psi(u_1',w_1',u_2',\ldots,w_3').$$

Mapping $\mathcal{R}_{123}$ looks complicated:

$$\mathbf{w}'_1 \;=\; \frac{\mathbf{w}_2}{\mathbf{w}_1\mathbf{w}_3^{-1} \;-\; \omega^{1/2}\,\mathbf{u}_3\mathbf{w}_3^{-1} \;+\; \kappa_3\;\mathbf{w}_2^{-1}\mathbf{u}_3}; \qquad etc.$$

We now represent $\mathbf{u}_i,\ \mathbf{w}_i,\ \mathbf{u}'_i,\ \mathbf{w}'_i$ by $N \times N$ matrices

$$\mathbf{u}_i \;=\; u_i\,\mathbf{X}_i; \qquad \mathbf{w}_i \;=\; w_i\,\mathbf{Z}_i; \qquad \mathbf{X}_i\,\mathbf{Z}_i = \omega\,\mathbf{Z}_i\,\mathbf{X}_i.$$

So $\exists$ a $N^3 \times N^3$-matrix $\mathbf{R}_{123}$ with

$$\frac{\mathbf{u}'_i}{u'_i} \;=\; \mathbf{R}_{123}\,\frac{\mathbf{u}_i}{u_i}\;\mathbf{R}_{123}^{-1}, \qquad \frac{\mathbf{w}'_i}{w'_i} \;=\; \mathbf{R}_{123}\,\frac{\mathbf{w}_i}{w_i}\;\mathbf{R}_{123}^{-1}, \qquad i{=}1,2,3.$$

We shall see: $\mathbf{R}_{123}$ can be written in simple form!

$\mathcal{R}_{123}$: *superposition of a functional mapping* $\mathcal{R}_{123}^{(f)}$

and a finite dimensional similarity transform $\mathbf{R}_{123}$:

$$\mathcal{R}_{123} \circ \Phi \;=\; \mathbf{R}_{123}\;\left(\mathcal{R}_{123}^{(f)} \circ \Phi\right)\;\mathbf{R}_{123}^{-1}\,.$$

This superposition because Weyl variables at root of unity!

(Bazhanov, Reshetikhin, Bobenko,
Sergeev, Mangazeev, Stroganov 1995)

$\mathbf{R}_{123}$ will be Boltzmann weights of N-component spin system.

Introduce Baxter-Bazhanov cyclic Fermat-curve functions:

related to quantum dilogarithm at root of unity: Faddeev-Kashaev 1993

$$n \in \mathbb{Z}_N\,, \qquad p = (x,\, y)\,, \qquad x^N + y^N = 1,$$

$$w_p(0) = 1, \qquad w_p(n) = \prod_{\nu=1}^{n} \frac{y}{1 - \omega^\nu x}$$

Because of Fermat relation: $w_p(n+N) = w_p(n)$.

Then:

$$R^{j_1,j_2,j_3}_{i_1,i_2,i_3} \equiv \langle i_1, i_2, i_3 | \mathbf{R}_{123} | j_1, j_2, j_3 \rangle$$
$$= \delta_{i_2+i_3,j_2+j_3}\, \omega^{(j_1-i_1)\, j_3}\, \frac{w_{p_1}(i_2-i_1)\, w_{p_2}(j_2-j_1)}{w_{p_3}(j_2-i_1)\, w_{p_4}(i_2-j_1)}$$

x-coord. of the four Fermat curve points are connected by

$$x_1\, x_2 = \omega\, x_3\, x_4\,.$$

Sergeev, Mangazeev, Stroganov 1995

Fermat curve points are defined in terms of $u_1, w_1, \kappa_1, \ldots, \kappa_3$:

$$x_1 = \frac{u_2}{\sqrt{\omega}\kappa_1 u_1}\,, \quad x_2 = \frac{\kappa_2 u_2'}{\sqrt{\omega} u_1'}\,, \quad x_3 = \frac{u_2'}{\omega u_1}\,,$$

Primes denote functionally transformed scalar variables:

$$u_j' = \mathcal{R}^{(f)}_{1,2,3} \circ u_j, \qquad w_j' = \mathcal{R}^{(f)}_{1,2,3} \circ w_j.$$

How to prove all this? Use recursion relations like

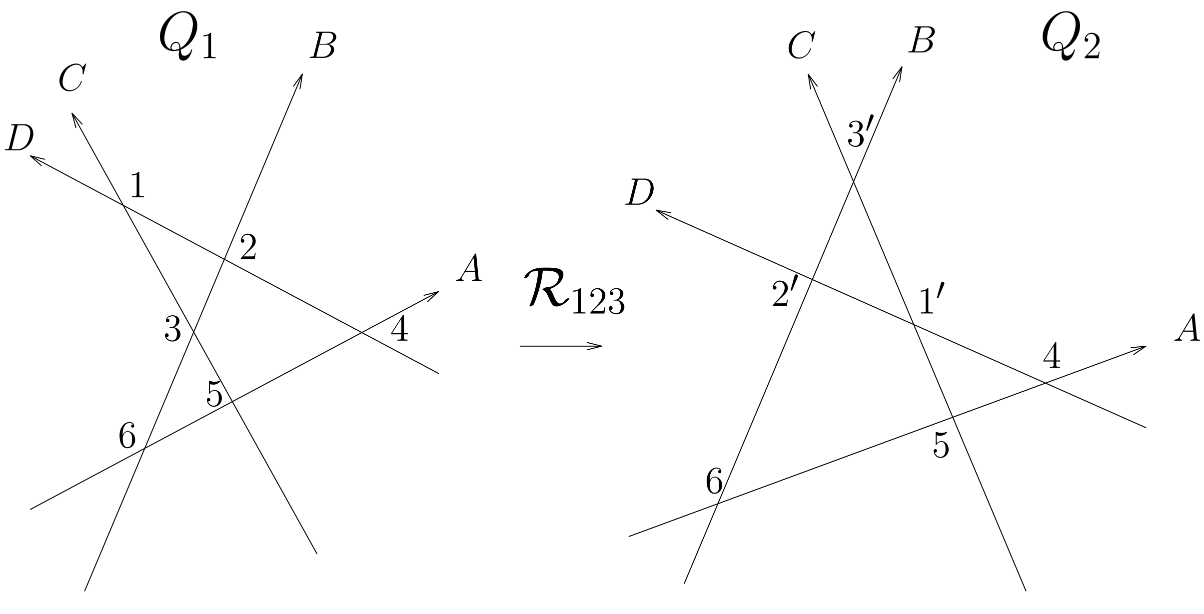
$$R^{j_1,j_2,j_3}_{i_1,i_2+1,i_3-1} = R^{j_1,j_2,j_3}_{i_1,i_2,i_3} \cdot \frac{y_1}{y_4} \cdot \frac{1 - \omega^{i_2-j_1+1} x_4}{1 - \omega^{i_2-i_1+1} x_1}\,.$$

The modified Tetrahedron equation.

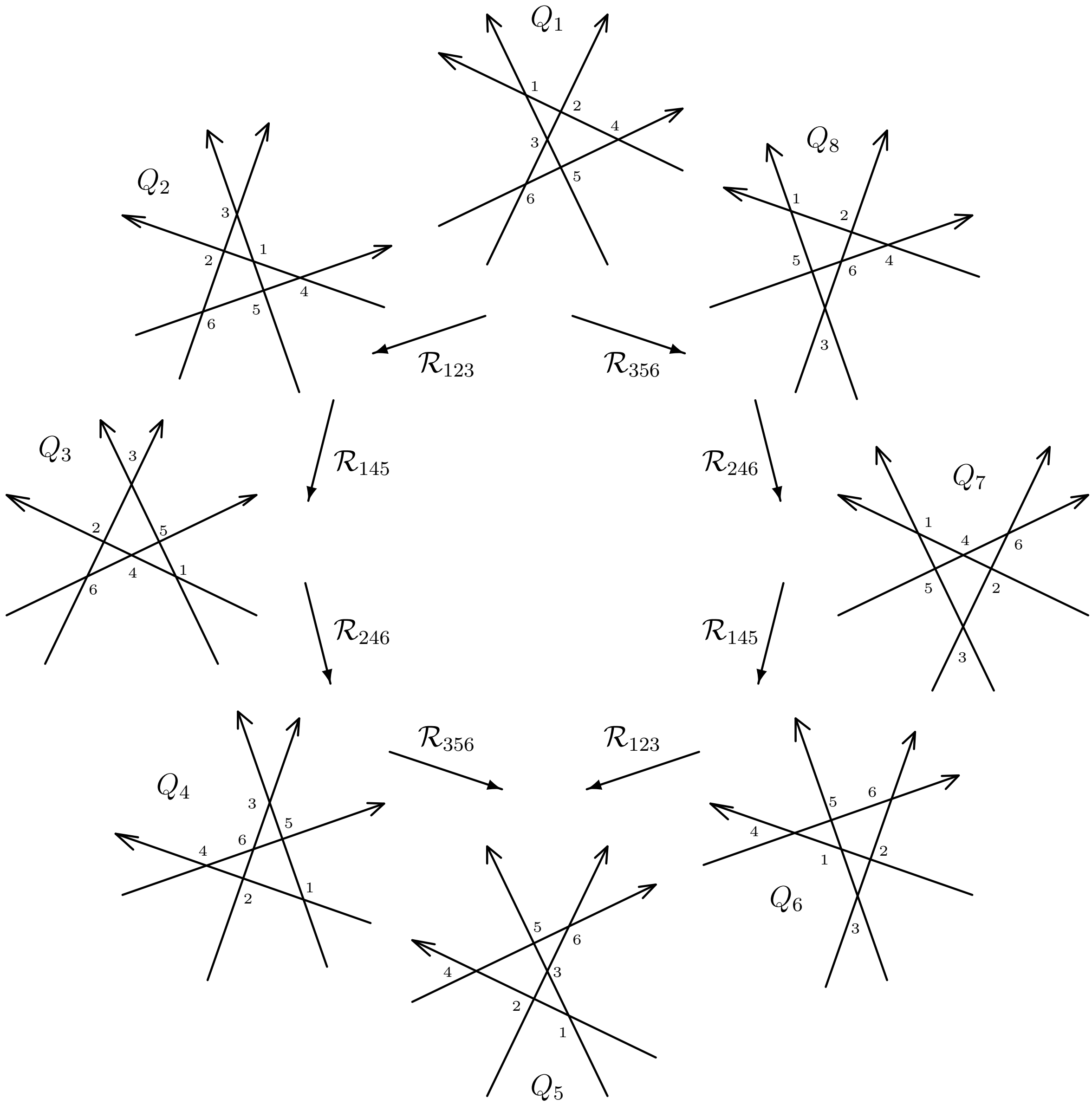
Basic 3-dim. lattice is formed by set of intersecting planes.

For Z-invariance $\Delta = \nabla$ we considered intersecting *3 planes*:
$Triangle$ in nearby cutting auxiliary plane,
shrinking to point at vertex $\mathcal{R}_{123}$.

Now take 4 intersecting planes of the basic lattice: These cut auxiliary plane in 4 lines forming $Quadrangle$ with 6 points.
Shifting one line such that one subtriangle Δ shrinks to point and then reverses to ∇: action of $\mathcal{R}_{ijk}$.
Each quadrangle has two subtriangles which can be reversed:
Figure Q_1 : we can use either $\mathcal{R}_{123}$ or $\mathcal{R}_{356}$:



$\exists$ 8 different quadrangles (with 6 labeled points)
$\Longrightarrow$ 8 different positions of the auxiliary plane w.r.t. vertices.



Each Q_i has *two* triangles which can be reversed

$\Rightarrow$ we can move only clock- or anticlockwise.

The clockwise and anticlockwise mappings $Q_1 \longrightarrow Q_5$

lead to the same result and so are equivalent:

$\Longrightarrow$ *The mapping* $\mathcal{R}$ *solves the TE:*

$$\mathcal{R}_{123} \cdot \mathcal{R}_{145} \cdot \mathcal{R}_{246} \cdot \mathcal{R}_{356} \; = \; \mathcal{R}_{356} \cdot \mathcal{R}_{246} \cdot \mathcal{R}_{145} \cdot \mathcal{R}_{123} \, .$$

Operator equation in Weyl space of $\mathbf{u}_1, \mathbf{w}_1, \mathbf{u}_2, \mathbf{w}_2, \ldots, \mathbf{u}_6, \mathbf{w}_6$.

TE for $\mathcal{R}_{ijk}$ $\Longrightarrow$ TE for $\mathcal{R}^{(f)}_{ijk}$:

$$\mathcal{R}^{(f)}_{123} \cdot \mathcal{R}^{(f)}_{145} \cdot \mathcal{R}^{(f)}_{246} \cdot \mathcal{R}^{(f)}_{356}\, \phi(u_1^N, \ldots, w_6^N) = \mathcal{R}^{(f)}_{356} \cdot \mathcal{R}^{(f)}_{246} \cdot \mathcal{R}^{(f)}_{145} \cdot \mathcal{R}^{(f)}_{123} \phi(u_1^N, \ldots, w_6^N).$$

Now we derive from the <u>Tetrahedron Equation</u>

$$\mathcal{R}_{123} \cdot \mathcal{R}_{145} \cdot \mathcal{R}_{246} \cdot \mathcal{R}_{356} = \mathcal{R}_{356} \cdot \mathcal{R}_{246} \cdot \mathcal{R}_{145} \cdot \mathcal{R}_{123}$$

the <u>*Modified Tetrahedron Equation*</u>.

We use $\mathcal{R}_{ijk} \circ \Phi = \mathbf{R}_{ijk} \left(\mathcal{R}^{(f)}_{ijk} \circ \Phi\right) \mathbf{R}^{-1}_{ijk}$ and get

$$\mathbf{R}_{123} \left[\mathcal{R}^{(f)}_{123} \left\{\mathbf{R}_{145} \left(\mathcal{R}^{(f)}_{145} \left[\mathbf{R}_{246} \left\{\mathcal{R}^{(f)}_{246} \left(\mathbf{R}_{356} \left(\mathcal{R}^{(f)}_{356} \circ \Phi\right) \mathbf{R}^{-1}_{356}\right)\right\} \mathbf{R}^{-1}_{246}\right]\right) \mathbf{R}^{-1}_{145}\right\}\right] \mathbf{R}^{-1}_{123}$$
$$= \mathbf{R}_{356} \left[\mathcal{R}^{(f)}_{356} \left\{\mathbf{R}_{246} \left(\mathcal{R}^{(f)}_{246} \left[\mathbf{R}_{145} \left\{\mathcal{R}^{(f)}_{145} \left(\mathbf{R}_{123} \left(\mathcal{R}^{(f)}_{123} \circ \Phi\right) \mathbf{R}^{-1}_{123}\right)\right\} \mathbf{R}^{-1}_{145}\right]\right) \mathbf{R}^{-1}_{246}\right\}\right] \mathbf{R}^{-1}_{356}.$$

Introduce shorthand:

$\mathbf{R}^{(1)} = \mathbf{R}_{123}; \quad \mathbf{R}^{(2)} = \mathcal{R}^{(f)}_{1,2,3} \circ \mathbf{R}_{145}; \quad \mathbf{R}^{(3)} = \mathcal{R}^{(f)}_{1,2,3}\mathcal{R}^{(f)}_{1,4,5} \circ \mathbf{R}_{246};$ *etc.*

giving

$$\left(\mathbf{R}^{(1)}\mathbf{R}^{(2)}\mathbf{R}^{(3)}\mathbf{R}^{(4)}\right)\left\{\mathcal{R}^{(f)}_{123} \cdot \mathcal{R}^{(f)}_{145} \cdot \mathcal{R}^{(f)}_{246} \cdot \mathcal{R}^{(f)}_{356}\, \Phi\right\}\left(\mathbf{R}^{(1)}\mathbf{R}^{(2)}\mathbf{R}^{(3)}\mathbf{R}^{(4)}\right)^{-1}$$
$$=\left(\mathbf{R}^{(8)}\mathbf{R}^{(7)}\mathbf{R}^{(6)}\mathbf{R}^{(5)}\right)\left\{\mathcal{R}^{(f)}_{356} \cdot \mathcal{R}^{(f)}_{246} \cdot \mathcal{R}^{(f)}_{145} \cdot \mathcal{R}^{(f)}_{123}\, \Phi\right\}\left(\mathbf{R}^{(8)}\mathbf{R}^{(7)}\mathbf{R}^{(6)}\mathbf{R}^{(5)}\right)^{-1}.$$

Functional TE cancels! We get the MTE :

$$\mathbf{R}^{(1)}\mathbf{R}^{(2)}\mathbf{R}^{(3)}\mathbf{R}^{(4)} = \rho\, \mathbf{R}^{(8)}\mathbf{R}^{(7)}\mathbf{R}^{(6)}\mathbf{R}^{(5)}$$

or, inserting the abbreviations:

$$\mathbf{R}_{123}\cdot\Big(\mathcal{R}^{(f)}_{123}\circ\mathbf{R}_{145}\Big)\cdot\Big(\mathcal{R}^{(f)}_{123}\mathcal{R}^{(f)}_{145}\circ\mathbf{R}_{246}\Big)\cdot\Big(\mathcal{R}^{(f)}_{123}\mathcal{R}^{(f)}_{145}\mathcal{R}^{(f)}_{246}\circ\mathbf{R}_{356}\Big)$$
$$=\ \rho\,\mathbf{R}_{356}\cdot\Big(\mathcal{R}^{(f)}_{356}\circ\mathbf{R}_{246}\Big)\cdot\Big(\mathcal{R}^{(f)}_{356}\mathcal{R}^{(f)}_{246}\circ\mathbf{R}_{145}\Big)\cdot\Big(\mathcal{R}^{(f)}_{356}\mathcal{R}^{(f)}_{246}\mathcal{R}^{(f)}_{145}\circ\mathbf{R}_{123}\Big)\ .$$

Modified Tetrahedron Equation.

$N^6\times N^6$ matrix equation with matrix entries

related by the functional transformation.

Not same rapidities left and right as e.g. in Yang-Baxter eq. :

Rapidities on left and right hand sides related by $\mathcal{R}^{(f)}_{ijk}$

which gives rise to classical integrable system

MTE contains 8 rapidity parameters and 16 phases.

Matrix structure:

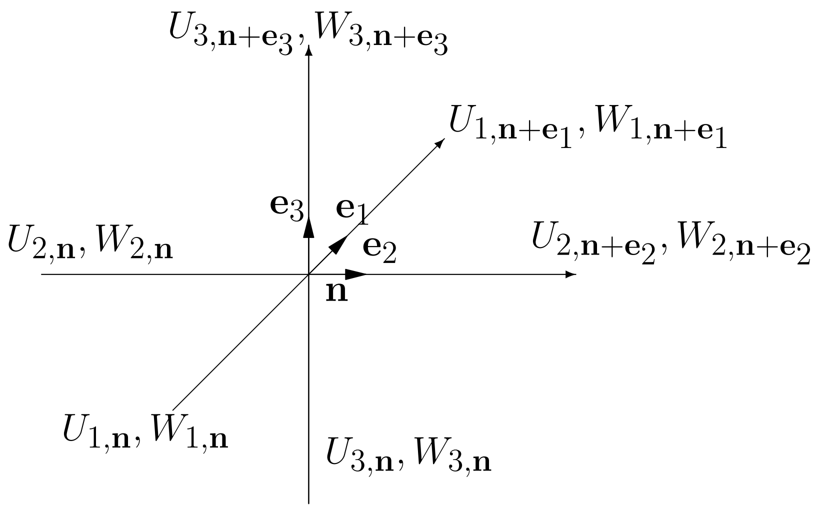
$$\mathbf{\Theta}\,^{k_1,k_2,k_3,k_4,k_5,k_6}_{i_1,i_2,i_3,i_4,i_5,i_6}\ =\ \rho\ \overline{\mathbf{\Theta}}\,^{k_1,k_2,k_3,k_4,k_5,k_6}_{i_1,i_2,i_3,i_4,i_5,i_6}$$

$\Rightarrow\ N^{12}$ eqs. (for $N=2$ 256 different non-trivial eqs.).

Different 3D models $\Longleftrightarrow$ Different solutions for functional mapping.

We obtain a large class of different 3D integrable quantum models choosing various classical solutions.

Sketch of functional mapping for whole 3D lattice, at site $\mathbf{n}$:



Recall:

$$w'^{N}_{1} \;=\; \frac{w_2^N}{w_1^N w_3^{-N} \;+\; u_3^N w_3^{-N} \;+\; \kappa_3^N w_2^{-N} u_3^N}\,; \qquad\qquad etc.$$

Define for the whole lattice: $\left(U = u^N,\; W = w^N,\; K = \kappa^N\right)$

$$\frac{U_{1,\mathbf{n}+\mathbf{e}_1}}{U_{1,\mathbf{n}}} = \frac{W_{3,\mathbf{n}+\mathbf{e}_3}}{W_{3,\mathbf{n}}}$$
$$= \frac{K_{2:n_1,n_3} U_{2,\mathbf{n}} W_{2,\mathbf{n}}}{K_{1:n_2,n_3} U_{1,\mathbf{n}} W_{2,\mathbf{n}} + K_{3:n_1,n_2} U_{2,\mathbf{n}} W_{3,\mathbf{n}} + K_{1:n_2,n_3} K_{3:n_1,n_2} U_{1,\mathbf{n}} W_{3,\mathbf{n}}}\,,$$

$$\frac{W_{1,\mathbf{n}}}{W_{1,\mathbf{n}+\mathbf{e}_1}} = \frac{W_{2,\mathbf{n}+\mathbf{e}_2}}{W_{2,\mathbf{n}}} = \frac{W_{1,\mathbf{n}} W_{3,\mathbf{n}}}{W_{1,\mathbf{n}} W_{2,\mathbf{n}} + U_{3,\mathbf{n}} W_{2,\mathbf{n}} + K_{3:n_1,n_2} U_{3,\mathbf{n}} W_{3,\mathbf{n}}}\,,$$

$$\frac{U_{2,\mathbf{n}+\mathbf{e}_2}}{U_{2,\mathbf{n}}} = \frac{U_{3,\mathbf{n}}}{U_{3,\mathbf{n}+\mathbf{e}_3}} = \frac{U_{1,\mathbf{n}} U_{3,\mathbf{n}}}{U_{2,\mathbf{n}} U_{3,\mathbf{n}} + U_{2,\mathbf{n}} W_{1,\mathbf{n}} + K_{1:n_2,n_3} U_{1,\mathbf{n}} W_{1,\mathbf{n}}}\,.$$

Rewrite this to trilinear Hirota form, we solve by methods of algebraic geometry (Fay-identities) (Krichever, Shiota, Mulase 1978-84)

Useful practical solution by rational limit of Θ-functions.

Trivial solution $\longrightarrow$ ZBB-model.

Conclusions:

We considered a 3-dim. *oriented* lattice with
affine Weyl variables at root of unity, located on links.
Studied *local* properties.

Linear current condition and $\Delta = \nabla$-invariance $\Rightarrow$
Sergeev rational canonical invertible mapping $\mathcal{R}_{123}$
acting on a triple affine Weyl algebra.

Mapping splits: functional mapping of Weyl centers and a
$N^3 \times N^3$ matrix conjugation:
Quantum operators with coeff. determined by classical integrable system.
Conjugation matrix represented by cyclic Fermat-curve functions $w_p(n)$.

TE from uniqueness of $\mathcal{R}$ $(Q_1 \Rightarrow Q_5)$,
MTE appears after cancellation of functional TE.

Explicit parameterization of MTE: 8 continous variables
vs. 5 in ZBB model. Describe phase transition?

Explicit solutions for functional mapping and generating
function for the constants of motion require
global considerations and boundary conditions.