# $sl_2$ Gaudin Model with Jordanian twist

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# Introduction

quantum spin systems

| Model | Quantum R-matrix | (dynamical) symmetry Algebra |
|---|---|---|
| XXX | rational | Yangian $Y(sl(2))$ |
| XXZ | trigonometric | quantum affine alg. $\mathcal{U}_q(\widehat{sl}(2))$ |
| XYZ | elliptic | Elliptic quantum gr. $\mathcal{E}_{\tau,\eta}(sl(2))$ |

- a class of quantum R-matrices, particular solutions of the Yang-Baxter equation

$$R_{12}(\lambda-\mu)\, R_{13}(\lambda-\nu)\, R_{23}(\mu-\nu) = R_{23}(\mu-\nu)\, R_{13}(\lambda-\nu)\, R_{12}(\lambda-\mu)$$

- the L operator $\qquad (V_a, z_a)$

$$L_{0a}(\lambda - z_a) = R_{0a}(\lambda - z_a) \qquad V_0 \text{ - aux.}$$

- the T-matrix

$$T(\lambda, \{z_a\}) = L_{0N}(\lambda - z_N) \cdots L_{01}(\lambda - z_1)$$

- Faddeev - Reshetikhin - Takhtajan (FRT) relations

$$R_{12}(\lambda-\mu)\, \underset{1}{T}(\lambda)\, \underset{2}{T}(\mu) = \underset{2}{T}(\mu)\, \underset{1}{T}(\lambda)\, R_{12}(\lambda-\mu)$$

- taking the trace over $V_1 \otimes V_2$ $\Rightarrow$ $t(\lambda) = \mathrm{tr}\, T$ generates an Abelian subalgebra

$$t(\lambda)\, t(\mu) = t(\mu)\, t(\lambda)$$

- Algebraic Bethe Ansatz yields the spectrum and the Bethe vectors

- semi-classical limit of the quantum spin system

$$R(\lambda;\eta) = I + \eta\, r(\lambda) + \mathcal{O}(\eta^2)$$

- Gaudin Hamiltonians are related to the classical r-matrix

$$H^{(a)} = \sum_{b \neq a} r_{ab}(z_a - z_b)$$

- commutativity $[H^{(a)}, H^{(b)}] = 0$ is guaranteed by the classical Yang-Baxter eq.:

$$[r_{ab}(z_a - z_b), r_{ac}(z_a - z_c) + r_{bc}(z_b - z_c)] + [r_{ac}(z_a - z_c), r_{bc}(z_b - z_c)] = 0$$

- substituting
$$R(\lambda;\eta) = I + \eta\, r(\lambda) + O(\eta^2)$$
$$T(\lambda;\eta) = I + \eta\, L(\lambda) + O(\eta^2)$$
into the FRT $\Rightarrow$

$$[\underset{1}{L}(\lambda), \underset{2}{L}(\mu)] = -[\underset{12}{r}(\lambda-\mu), \underset{1}{L}(\lambda) + \underset{2}{L}(\mu)]$$

the Sklyanin bracket

- a generating function of the Gaudin Hamiltonians is $t(\lambda) = \frac{1}{2}\, \mathrm{tr}\, L^2(\lambda)$

- $t(\lambda)$ generates an Abelian subalgebra

$$t(\lambda)\, t(\mu) = t(\mu)\, t(\lambda)$$

notice

$$\left[ t(\lambda), L(\mu) \right] = \left[ M(\lambda-\mu), L(\mu) \right],$$

$$M(\lambda-\mu) = - \underset{1}{\mathrm{tr}} \left( r_{12}(\lambda-\mu) \, \underset{1}{L}(\lambda) \right) - \frac{1}{2} \underset{1}{\mathrm{tr}} \left( r_{12}^2(\lambda-\mu) \right)$$

- a Gaudin realization: to every point $z_a$, $a = 1, 2, \dots, N$ correspond an irreducible representation $V_a^{(l_a)}$ of the $sl_2$

$$h_a \omega_a = l_a \omega_a$$

$$X_a^+ \omega_a = 0$$

then

$$\mathcal{H} = V_1^{(l_1)} \otimes \dots \otimes V_N^{(l_N)}$$

and

$$\Omega_+ = \omega_1 \otimes \dots \otimes \omega_N$$

- $L$-operator

$$L(\lambda) = \begin{pmatrix} h(\lambda) & 2X^-(\lambda) \\ 2X^+(\lambda) & -h(\lambda) \end{pmatrix}$$

where $h(\lambda) = \sum_{a=1}^{N} \frac{h_a}{\lambda - z_a}$, $X^\pm(\lambda) = \sum_{a=1}^{N} \frac{X_a^\pm}{\lambda - z_a}$.

- Then

$$h(\lambda)\,\Omega_+ = \rho(\lambda)\,\Omega_+ \,, \quad X^+(\lambda)\,\Omega_+ = 0$$

$$\rho(\lambda) = \sum_{a=1}^{N} \frac{l_a}{\lambda - z_a}$$

- the Gaudin Hamiltonians are the residues of $t(\lambda)$ at $z_a$, $a=1,\dots,N$

$$t(\lambda) = \sum_{a=1}^{N} \left( \frac{l_a(l_a+2)}{(\lambda - z_a)^2} + 2\,\frac{H^a}{\lambda - z_a} \right)$$

- notice

$$t(\lambda)\,\Omega_+ = \Lambda_0(\lambda)\,\Omega_+ ,$$

$$\Lambda_0(\lambda) = \rho^2(\lambda) - 2\rho'(\lambda) = \sum_{a=1}^{N} \frac{l_a(l_a+2)}{(\lambda - z_a)^2} + 2\sum_{a=1}^{N} \frac{1}{\lambda - z_a} \times \left( \sum_{b\neq a}^{N} \frac{l_a l_b}{z_a - z_b} \right)$$

- Algebraic Bethe Ansatz

$$\Psi(\mu_1, \dots, \mu_M) = X^-(\mu_1) \cdots X^-(\mu_M)\,\Omega_+$$

$$t(\lambda)\,\Psi(\mu_1, \dots, \mu_M) = \Lambda_M(\lambda; \{\mu_j\}_{j=1}^{M})\,\Psi(\mu_1, \dots, \mu_M)$$

$$\Lambda_M(\lambda; \{\mu_j\}_{j=1}^{M}) = \rho_M^2(\lambda; \{\mu_j\}) - 2\,\partial_\lambda \rho_M(\lambda; \{\mu_j\}),$$

$$\rho_M(\lambda; \{\mu_j\}) = \rho(\lambda) - \sum_{i=1}^{M} \frac{2}{\lambda - \mu_i},$$

once the Bethe equations are imposed on $\mu_1, \dots, \mu_M$

$$\rho_M(\mu_i; \mu_1, \dots, \mu_{i-1}, \mu_{i+1}, \mu_M) = \sum_{a=1}^{N} \frac{l_a}{\mu_i - z_a} - \sum_{j\neq i}^{M} \frac{2}{\mu_i - \mu_j} = 0$$

- the Bethe vectors are the eigenvectors of the Gaudin Hamiltonians

$$H^{(a)}\,\Psi(\mu_1,\dots,\mu_M) = E_M^{(a)}\,\Psi(\mu_1,\dots,\mu_M),$$

$$E_M^{(a)} = \sum_{b\neq a}^{N} \frac{l_a l_b}{z_a - z_b} - \sum_{i=1}^{M} \frac{2 l_a}{z_a - \mu_i},$$

once the Bethe equations are imposed on $\mu_1,\dots,\mu_M$.

# Twists

- quasitriangular Hopf algebra

$\mathcal{A}(m, 1, \Delta, \epsilon, S; \mathcal{R})$

- multiplication

$m: \mathcal{A}\otimes\mathcal{A} \to \mathcal{A}$

$m(a\otimes b) = a\cdot b$

$$(a\cdot b)\cdot c = a\cdot(b\cdot c)$$
$$a\cdot 1 = 1\cdot a = a$$
$$a\cdot(\alpha b + \beta c) = \alpha\, a\cdot b + \beta\, a\cdot c$$
$$(\alpha a + \beta b)\cdot c = \alpha\, a\cdot c + \beta\, b\cdot c$$

- coproduct

$\Delta: A \longrightarrow \mathcal{A}\otimes\mathcal{A}$

$$(\Delta\otimes id)(\Delta(a)) = (id\otimes\Delta)(\Delta(a))$$
$$(id\otimes\epsilon)(\Delta(a)) = (\epsilon\otimes id)(\Delta(a))$$

$\epsilon: \mathcal{A} \to \mathbb{C}$

$$\Delta(a\cdot b) = \Delta(a)\cdot\Delta(b)$$
$$\epsilon(a\cdot b) = \epsilon(a)\cdot\epsilon(b)$$

- antipode $S: \mathcal{A} \longrightarrow \mathcal{A}$

$$m\circ(S\otimes id)(\Delta(a)) = \epsilon(a)1$$
$$m\circ(id\otimes S)(\Delta(a)) = \epsilon(a)1$$
$$S(a\cdot b) = S(b)\cdot S(a)$$

- commutative $\quad a \cdot b = b \cdot a \qquad m = m \circ \sigma$

  cocommutative $\quad \Delta(a) = \sum_i a_i \otimes b_i = \sum_i b_i \otimes a_i = \Delta'(a) \qquad \Delta' = \sigma \circ \Delta$

- universal R-matrix $\qquad \mathcal{R} \in \mathcal{A} \otimes \mathcal{A}$

$$\mathcal{R}\, \Delta(a) = \Delta'(a)\, \mathcal{R}$$

$$(\Delta \otimes id)(\mathcal{R}) = \mathcal{R}_{13} \mathcal{R}_{23} \qquad \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$$

$$(id \otimes \Delta)(\mathcal{R}) = \mathcal{R}_{13} \mathcal{R}_{12}$$

$$\Rightarrow \mathcal{R}_{12} \mathcal{R}_{13} \mathcal{R}_{23} = \mathcal{R}_{23} \mathcal{R}_{13} \mathcal{R}_{12}$$ Yang-Baxter equation

$$(S \otimes id)(\mathcal{R}) = \mathcal{R}^{-1} = (id \otimes S^{-1})(\mathcal{R})$$

$$(S \otimes S)(\mathcal{R}) = \mathcal{R}$$

$$(\epsilon \otimes id)(\mathcal{R}) = 1 = (id \otimes \epsilon)(\mathcal{R})$$

- twist as a similarity transformation of the coproduct

$$\Delta_t(a) = \mathcal{F}\,\Delta(a)\,\mathcal{F}^{-1}, \qquad \mathcal{F} \in \mathcal{A}\otimes\mathcal{A}$$

to preserve the axioms of a Hopf algebra

$$(\epsilon \otimes id)(\mathcal{F}) = (id \otimes \epsilon)(\mathcal{F}) = 1$$

$$\mathcal{F}_{12}\,(\Delta \otimes id)(\mathcal{F}) = \mathcal{F}_{23}\,(id \otimes \Delta)(\mathcal{F}) \qquad \in \mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A}$$

twist eq.

coassociativity of $\Delta_t$

antipode is given by

$$S_t(a) = v\,S(a)\,v^{-1}, \qquad v = \sum_i f_i^{(1)} S(f_i^{(2)})$$

$$\mathcal{F} = \sum_i f_i^{(1)} \otimes f_i^{(2)}$$

$\Rightarrow \mathcal{A}_t(m, 1, \Delta_t, \epsilon, S_t)$ is a Hopf algebra

- let $\mathcal{A}(m, 1, \Delta, \epsilon, S; \mathcal{R})$ be a quasitri

$$\Delta'(a) = \mathcal{R}\Delta(a)\mathcal{R}^{-1}$$

then $\mathcal{A}_t(m, 1, \Delta_t, \epsilon\ S_t; \mathcal{R}_t)$ has the universal $\mathcal{R}$-matrix

$$\mathcal{R}_t = \mathcal{F}_{21}\, \mathcal{R}\, \mathcal{F}_{12}^{-1}$$

- an importan subclass of factorizabl twists consists of elements satisfying

$$(\Delta \otimes id)(\mathcal{F}) = \mathcal{F}_{13}\, \mathcal{F}_{23}$$
$$(id \otimes \Delta_t)(\mathcal{F}) = \mathcal{F}_{12}\, \mathcal{F}_{13}$$

notice that $\mathcal{R}$ satisfies these equations for $\Delta_t = \Delta'$.

$\{h_i\}_1^n$ are mutually commuting and primitive

$[h_i, h_j] = 0$ $\quad \Delta(h_i) = h_i \otimes 1 + 1 \otimes$

Reshetikhin $\quad \Rightarrow \mathcal{F} = \exp\left(\sum_{i,j=1}^{n} \varphi_{ij}\, h_i \otimes h_j\right)$

- Jordanian twist of $sl_2$

$$\mathcal{F}^{J} = \exp(h \otimes \sigma) = \exp\left(\tfrac{1}{2} h \otimes \ln(1+2\theta X)\right)$$

due to the
$$\Delta(h) = h \otimes 1 + 1 \otimes h$$
$$\Delta_t(\sigma) = \sigma \otimes 1 + 1 \otimes \sigma$$

$$(\Delta \otimes id)(e^{h\otimes\sigma}) = e^{h\otimes 1\otimes\sigma}\, e^{1\otimes h\otimes\sigma},$$
$$(id \otimes \Delta_t)(e^{h\otimes\sigma}) = e^{h\otimes\sigma\otimes 1}\, e^{h\otimes 1\otimes\sigma}.$$

- to twist the spin system twist the Yang R-matrix

$$R_{tw}(\lambda) = \mathcal{F}^{J}_{21}\left(\mathbb{1} + \frac{\eta}{\lambda}\mathcal{P}\right)\mathcal{F}^{J\,-1}_{12}$$

$$R_{tw}(\lambda) = \begin{pmatrix} 1+\frac{\eta}{\lambda} & -\theta & \theta & \theta^2 \\ 0 & 1 & \frac{\eta}{\lambda} & -\theta \\ 0 & \frac{\eta}{\lambda} & 1 & \theta \\ 0 & 0 & 0 & 1+\frac{\eta}{\lambda} \end{pmatrix}$$

- $$T(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}, \qquad FRT \Rightarrow$$

$$\left(1 + \frac{\eta}{\lambda-\mu}\right) A(\lambda) A(\mu) - \theta A(\lambda) C(\mu) + \theta C(\lambda) A(\mu) +$$
$$+ \theta^2 C(\lambda) C(\mu) = \left(1 + \frac{\eta}{\lambda-\mu}\right) A(\mu) A(\lambda)$$

do not commute

- . . .

$\Rightarrow$ $B$'s do not commute, $D$'s also

and $\quad BD \neq DB$

but $\quad C(\lambda) C(\mu) = C(\mu) C(\lambda)$

- let $\Omega_+ = \omega_1 \otimes \dots \otimes \omega_N$

$$C(\lambda)\, \Omega_+ = 0$$
$$\Rightarrow \quad A(\lambda)\, \Omega_+ = a(\lambda)\, \Omega_+$$
$$D(\lambda)\, \Omega_+ = d(\lambda)\, \Omega_+$$

just like [illegible]
it $H_z$ [illegible]
$\theta = 0$

- the central element

$$\det_{q,\theta}(T(\lambda)) = a(\lambda + \eta/2)\, d(\lambda - \eta/2)$$

the same as in the case $\theta=$

- Algebraic Bethe Ansatz

$$t(\lambda)\, \Psi(\mu_1, \dots, \mu_M) = \Lambda(\lambda; \{\mu_j\}_1^M)\, \Psi(\mu_1, \dots$$

the same as in the case $\theta=0$

more complicated since
$B(\mu)\, B(\nu) \neq B(\nu)\, B(\mu)$ when $\theta \neq 0$

- the Gaudin model as the semiclassica[l] limit $\theta = -\frac{\eta}{2}\varepsilon$, $\eta \to 0$

- Kulish noticed that the similarity transformation by $e^{\alpha X^+} \otimes e^{\alpha X^+}$ on the $sl_2$ trigonometrical r-matrix

$$r_{trig}(\lambda) = \frac{e^{\lambda}}{sh(\lambda)} r_{DJ} + \frac{e^{-\lambda}}{sh(\lambda)} (r_{DJ})_{21},$$

$r_{DJ}$ is the Drinfeld – Jimbo constant r-matrix,

setting $\lambda \to \epsilon\lambda$, $\alpha \to \frac{\xi}{2\epsilon}$ after the scaling limit

$$\lim_{\epsilon \to 0} \epsilon\, r_{trig}(\epsilon\lambda) = \frac{1}{\lambda}\left(h \otimes h + 2(X^+ \otimes X^- + X^- \otimes X^+)\right) + \xi\left(h \otimes X^+ - X^+ \otimes h\right)$$

yields the $sl_2$ - inv. r-matrix deformed by the constant Jordanian r-matrix.

- Moreover $e^{\alpha\left(\sum_{a=1}^{N} X_a^+\right)}\,\Omega_+ = \Omega_+$

- Based on these arguments Kulish postulated the Bethe vectors, the spectrum and the Bethe equations:

$$\Psi_M(\mu_1,\dots,\mu_M) = X^-(\mu_1)\left(X^-(\mu_2)+\xi\right)\cdots\left(X^-(\mu_M)+(M-1)\xi\right)\Omega_+$$

the spectrum and Bethe eq. are the same as in the case $\xi = 0$.

# TWISTED GAUDIN MODEL

- the classical r-matrix

$$r(\lambda) = \frac{C_2^{\otimes}}{\lambda} + \xi\, r_J = \begin{pmatrix} \frac{1}{\lambda} & \xi & -\xi & 0 \\ 0 & -\frac{1}{\lambda} & \frac{2}{\lambda} & \xi \\ 0 & \frac{2}{\lambda} & -\frac{1}{\lambda} & -\xi \\ 0 & 0 & 0 & \frac{1}{\lambda} \end{pmatrix}$$

and

$$L(\lambda) = \begin{pmatrix} h(\lambda) & 2X^-(\lambda) \\ 2X^+(\lambda) & -h(\lambda) \end{pmatrix}$$

- the Sklyanin bracket

$$\left[ \underset{1}{L}(\lambda), \underset{2}{L}(\mu) \right] = -\left[ r_{12}(\lambda-\mu), \underset{1}{L}(\lambda) + \underset{2}{L}(\mu) \right]$$

$$\Rightarrow \quad [h(\lambda), h(\mu)] = 2\xi\,(X^+(\lambda) - X^+(\mu))$$

$$[X^-(\lambda), X^-(\mu)] = -\xi\,(X^-(\lambda) - X^-(\mu))$$

$$[X^+(\lambda), X^-(\mu)] = -\frac{h(\lambda) - h(\mu)}{\lambda-\mu} + \xi X^+(\lambda)$$

$$[X^+(\lambda), X^+(\mu)] = 0$$

$$[h(\lambda), X^-(\mu)] = 2\,\frac{X^-(\lambda) - X^-(\mu)}{\lambda-\mu} + \xi h(\mu)$$

$$[h(\lambda), X^+(\mu)] = -2\,\frac{X^+(\lambda) - X^+(\mu)}{\lambda-\mu}$$

- the Gaudin realization

$$L(\lambda) = \sum_{a=1}^{N}\left(\frac{1}{\lambda-z_a}\begin{pmatrix} h_a & 2X_a^- \\ 2X_a^+ & -h_a\end{pmatrix} + \xi\begin{pmatrix} X_a^+ & -h_a \\ 0 & -X_a^+\end{pmatrix}\right)$$

- also

$$X^+(\lambda)\Omega_+ = 0, \quad h(\lambda)\Omega_+ = \rho(\lambda)\Omega_+, \quad \rho(\lambda) = \sum_{a=1}^{N}\frac{\ell_a}{\lambda-z_a}$$

- $$t(\lambda) = \tfrac{1}{2}\,\mathrm{tr}\, L^2(\lambda) = h^2(\lambda) + 2\big(X^+(\lambda)X^-(\lambda) + X^-(\lambda)X^+(\lambda)\big)$$
$$= h^2(\lambda) - 2h'(\lambda) + 2\big(2X^-(\lambda) + \xi\big)X^+(\lambda)$$

has the expansion

Casimir at $a$

$$t(\lambda) = \sum_{a=1}^{N}\left(\frac{C_2(a)}{(\lambda - z_a)^2} + \frac{2H^{(a)}}{\lambda - z_a}\right) + \xi^2 \sum_{a,b} X_a^+ X_b^+ ,$$

$$H^{(a)} = \sum_{b \neq a}^{N}\left(\frac{C_2^{\otimes}(a,b)}{z_a - z_b} + \xi\left(h_a X_b^+ - X_a^+ h_b\right)\right)$$

- to do ABA

$$t(\lambda)\,\Omega_+ = \left(h^2(\lambda) - 2h'(\lambda)\right)\Omega_+ = \Lambda_0(\lambda)\,\Omega_+$$ ✓

just [illegible]

$$t(\lambda)\,X^-(\mu)\Omega_+ = X^-(\mu)\,t(\lambda)\Omega_+ + \left[t(\lambda), X^-(\mu)\right]\Omega_+$$

the commutator is very different from the case $\xi = 0$, but

$$t(\lambda)\,X^-(\mu)\Omega_+ = \Lambda_1(\lambda;\mu)\,X^-(\mu)\Omega_+ \quad \text{when } \varrho(\mu) = 0 \;!$$

- however in general

$$\psi_M(\mu_1,\dots,\mu_M) = B_M(\mu_1,\dots,\mu_M)\,\Omega_+$$
$$= X^-(\mu_1)\left(X^-(\mu_2)+\xi\right)\cdots\left(X^+(\mu_3)+(M-1)\xi\right)$$

$B_M(\mu_1,\dots,\mu_M)$ are symmetric functions

- to get the spectrum we need [illegible]

$$t(\lambda)\,B_M(\vec{\mu}) = B_M(\vec{\mu})\left(t(\lambda)-\sum_{i=1}^{M}\frac{4h(\lambda)}{\lambda-\mu_i}+\sum_{i<j}^{M}\frac{8}{(\lambda-\mu_i)(\lambda-\mu_j)}\right)$$
$$+4\sum_{i=1}^{M}\frac{B_M(\vec{\mu}^{(i)}\cup\lambda)}{\lambda-\mu_i}\left(h(\mu_i)-\sum_{j\neq i}^{M}\frac{2}{\mu_i-\mu_j}\right)$$
$$+2\xi\sum_{i=1}^{M}B^{(1)}_{M-1}(\vec{\mu}^{(i)})\,h(\lambda)\left(h(\mu_i)-\sum_{j\neq i}^{M}\frac{2}{\mu_i-\mu_j}\right)$$
$$+4\xi\sum_{i\neq j}\frac{B^{(1)}_{M-1}(\lambda\cup\vec{\mu}^{(i,j)})-B^{(1)}_{M-1}(\vec{\mu}^{(i)})}{\lambda-\mu_j}\left(h(\mu_i)-\sum_{k\neq i}\frac{2}{\mu_i-\mu_k}\right)$$
$$+\xi^2\sum_{i\neq j}B^{(2)}_{M-2}(\vec{\mu}^{(i,j)})\;\hat{\beta}_{M-1}(\mu_j;\vec{\mu}^{(i,j)})\,\hat{\beta}_M(\mu_i;\mu^{(i)})$$
$$+4M\xi B_M(\vec{\mu})\,X^+(\lambda)+2\xi^2\sum_{i=1}^{M}B^{(1)}_{M-1}(\vec{\mu}^{(i)})\,X^+(\mu_i)$$

$\Rightarrow$ Yes, the spectrum is the same as in the case $\xi = 0$, once the Bethe equation are imposed the additional terms are $= 0$ !

$$t(\lambda)\,\Psi_M(\mu_1,\dots,\mu_M) = \Lambda(\lambda;\{\mu_j\}_1^M)\,\Psi_M(\mu_1,\dots,\mu_M)$$

$$\Lambda_M(\lambda;\{\mu_j\}_1^M) = \Lambda_0(\lambda) - \sum_{i=1}^{M} \frac{4\rho(\lambda)}{\lambda-\mu_i} + \sum_{i<j} \frac{8}{(\lambda-\mu_i)(\lambda-\mu_j)}$$

once $$\sum_{a=1}^{N} \frac{l_a}{\mu_i - z_a} - \sum_{j\neq i} \frac{2}{\mu_i-\mu_j} = 0 \quad , \quad i = 1,\dots,M.$$

also $$H^{(a)}\,\Psi_M(\mu_1,\dots,\mu_M) = E_M^{(a)}\,\Psi_M(\mu_1,\dots,\mu_M)$$

$$E_M^{(a)} = \sum_{b\neq a} \frac{l_a l_b}{z_a - z_b} - \sum_{i=1}^{M} \frac{2 l_a}{z_a - \mu_i}$$

- also $X^+_{ge}\,\psi_M(\mu_1,\dots,\mu_M) = 0$ ✓

once the Bethe eq. are imposed

but
$$h_{ge}\,\psi_M(\mu_1,\dots,\mu_M) = -2\,p_1^{(M)}\,\psi_M(\mu_1,\dots,\mu_M) +$$
$$+\,2\xi\,p_1^{(M-1)}\sum_{i=1}^{M}\psi_{M-1}(\mu_1,\dots,\mu_{i-1},\mu_{i+1}\dots$$

Bethe vectors are not the eigenstates of the [illegible] eq.!

- the B-operators satisfy the following identity

$$\partial_{z_a} B_M(\mu_1,\dots,\mu_M) = -\sum_{i=1}^{M}\partial_{\mu_i}\Big(X_a^-(\mu_i)\,B^{(1)}_{M-1}(\mu_1,\dots,\mu_{i-1},\mu_{i+1}\dots$$

which is important when solving the KZ-equations.

# Conclusions

- The $sl_2$ Gaudin model with Jordanian twist has the same spectrum as the invariant model although the Bethe states are different.

- The dual B-operators are used to optain the inner products and the norms of the Bethe states.

- The relation between the Bethe vectors and the solution to the KZ-equation is analogous to the invariant case.