Entanglement Entropy and Quantum Field Theory

Pasquale Calabrese

Oxford University

Outline:

- Entanglement Entropy in QFT
- Path integral formula for the entropy
- Exact calculations with CFT in 1+1 dimensions
- Non critical $1 + 1-$dimensional systems
- Unitary dynamics of entanglement

[P. Calabrese and J. Cardy, hep-th/0405152]
[cond-mat/0503393]

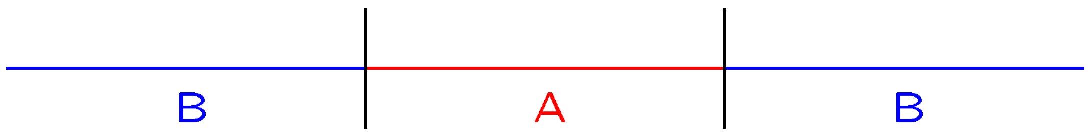
London 23/04/2005

Entanglement Entropy and QFT

Quantum system in the ground state $|\Psi\rangle$

The density matrix is $\rho = |\Psi\rangle\langle\Psi|$ ($\text{Tr}\rho = 1$)

A measures a subset, B the remainder:



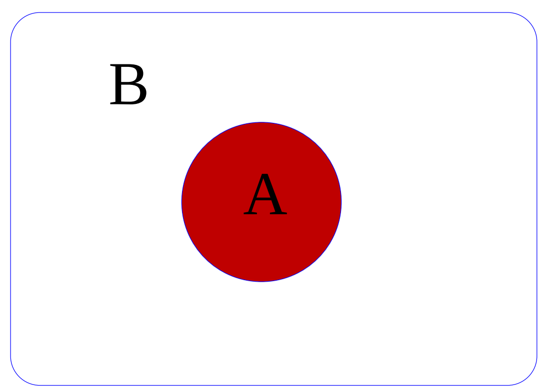
Reduced density matrix $\rho_A = \text{Tr}_B\rho$ ($\rho_B = \text{Tr}_A\rho$)

Entanglement Entropy $\equiv$ Von Neumann entropy of ρ_A:

$$S_A = -\text{Tr}\,\rho_A \ln \rho_A$$

[note $S_A = S_B$]

Historical review:



- Srednicki '93: Area Law
in a $d+1$ critical $T=0$ QFT
$S_A \propto \mathcal{A} \Rightarrow S \propto \mathcal{A}\Lambda^{d-1}$
and for $d=1$?
$S \propto \ln\Lambda \Rightarrow S \propto \ln \ell\Lambda$

Non extensive

- Holzhey, Larsen, Wilczek '94: In a 1+1D $T=0$ CFT

$$S_A = \frac{c}{3}\ln\frac{\ell}{a}$$

Entropy and path integral

Lattice QFT in 1+1 dimensions

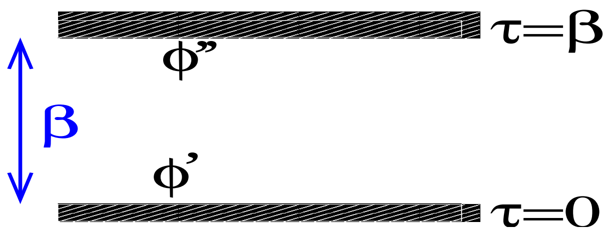
$\{\hat{\phi}(x)\}$ a set of fundamental fields with eigenvalues $\{\phi(x)\}$ and eigenstates $\otimes_x|\{\phi(x)\}\rangle$

The density matrix at temperature β^{-1} is

$$\rho(\{\phi(x'')''\}|\{\phi(x')'\}) = Z^{-1}\langle\{\phi(x'')''\}|e^{-\beta\hat{H}}|\{\phi(x')'\}\rangle$$

$Z = \mathrm{Tr}\, e^{-\beta\hat{H}}$ is the partition function.

Euclidean path integral:

$\rho =$ 

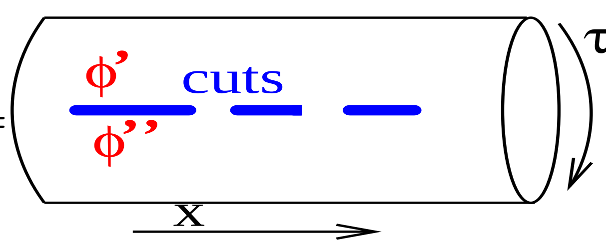
$=$

$$\frac{1}{Z}\int [d\phi(x,\tau)] \prod_x \delta(\phi(x,0)-\phi(x')') \prod_x \delta(\phi(x,\beta)-\phi(x'')'')\, e^{-S_E}$$

$S_E = \int_0^\beta L_E d\tau$, with L_E the Euclidean Lagrangian

The trace has the effect of sewing together the edges along $\tau = 0$ and $\tau = \beta$ to form a cylinder of circumference β.

$A = (u_1, v_1), \ldots, (u_N, v_N)$: ρ_A sewing together only those points x which are not in A. This will have the effect of leaving open cuts, one for each interval (u_j, v_j), along the the line $\tau = 0$.

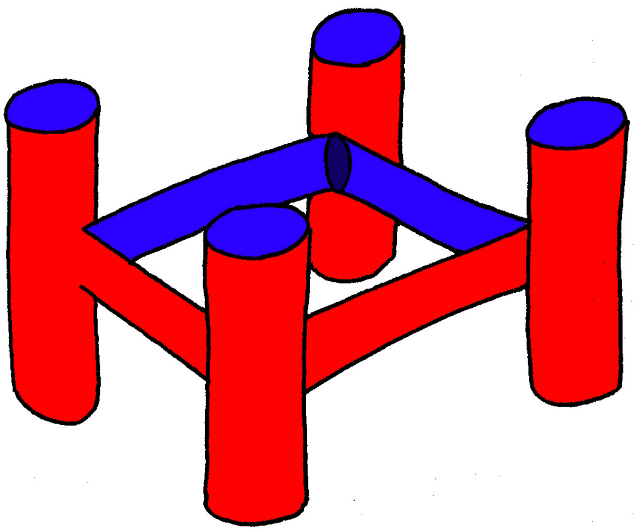
$\rho_A =$ 

$= \int_{x\in B} [d\phi(x,0)]\delta(\phi(x,\beta)-\phi(x,0))\rho$

“Replica trick”

$$S_A = -\text{Tr}\rho_A \log \rho_A = -\lim_{n\to 1} \frac{\partial}{\partial n} \text{Tr}\rho_A^n$$

$\text{Tr}\rho_A^n$ (for integer n) is the partition function on n of the above cylinders attached to form an $n-$sheeted Riemann surface



$$= \text{“}\rho_A^{ij}\rho_A^{jk}\rho_A^{kl}\rho_A^{li}\text{”}$$

$\text{Tr}\rho_A^n$ has a unique analytic continuation to $\text{Re}\, n > 1$ and that its first derivative at $n = 1$ gives the required entropy:

$$S_A = -\lim_{n\to 1} \frac{\partial}{\partial n} \frac{Z_n(A)}{Z^n}$$

Continuum limit: $a \to 0$ [Most of UV div cancel in the ratio]

Entropy and CFT

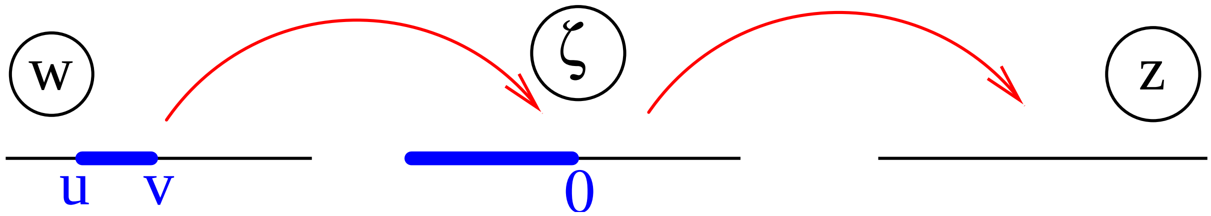
0. Single interval (u,v). We need $Z_n/Z^n = \langle 0|0\rangle_{\mathcal{R}_n}$. Thus we have to compute $\langle T(w)\rangle_{\mathcal{R}_n}$

Under a conformal transformation $w \to z$

$$T(w) = \left(\frac{dz}{dw}\right)^2 T(z) + \frac{c}{12}\frac{z'''z' - 3/2 z''^2}{z'^2}$$

Thus

$$w \to \zeta = \tfrac{w-u}{w-v};\ \zeta \to z = \zeta^{1/n} \Rightarrow\ w \to z = \left(\tfrac{w-u}{w-v}\right)^{1/n}$$



But $\langle T(z)\rangle_{\mathrm{C}} = 0 \Rightarrow$

$$\langle T(w)\rangle_{\mathcal{R}_n} = \frac{c(1-(1/n)^2)}{24}\frac{(v-u)^2}{(w-u)^2(w-v)^2}$$

To be compared with the Conformal Ward identities:

$$\frac{\langle T(w)\Phi_n(u)\Phi_{-n}(v)\rangle_{\mathrm{C}}}{\langle \Phi_n(u)\Phi_{-n}(v)\rangle_{\mathrm{C}}} = \frac{\Delta_\Phi (v-u)^2}{(w-u)^2(w-v)^2}$$

$$\Downarrow$$

Z_n/Z^n transforms under conformal transformations (acting identically on each sheet) as nth power of the two point function of a (fake) primary field on the complex plane with scaling dimension

$$\boxed{\Delta_\Phi = \overline{\Delta}_\Phi = \frac{c}{24}\left(1 - \frac{1}{n^2}\right)}$$

Recall that $\langle \phi(x)\phi(y)\rangle = |x-y|^{-4\Delta_\Phi}$

$$\mathrm{Tr}\,\rho_A^n = \frac{Z_n}{Z^n} = c_n\left(\frac{v-u}{a}\right)^{-(c/6)(n-1/n)}$$

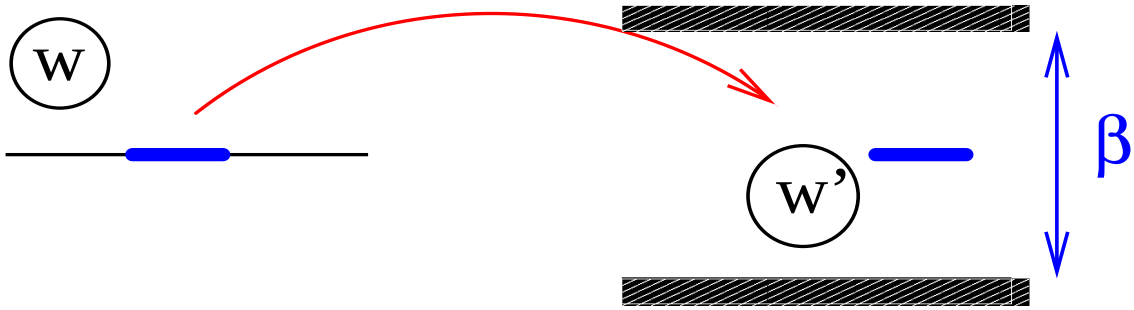
Finally with the replica trick $(v-u=\ell)$

$$S_A = \frac{c}{3}\ln\frac{\ell}{a} + c_1'$$

Generalizations

1. Finite temperature: map the plane into a cylinder

$$w \to w' = \frac{\beta}{2\pi} \ln w$$



$$S_A(\beta) \sim \frac{c}{3} \log \left(\frac{\beta}{\pi a} \sinh \frac{\pi \ell}{\beta} \right) + c_1' \,.$$

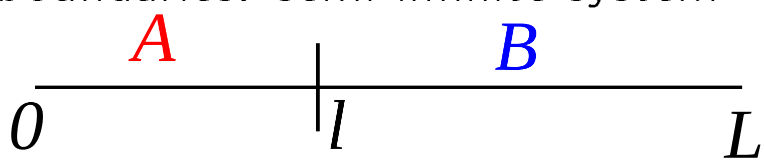
$$S_A \simeq \begin{cases} \dfrac{\pi c}{3} \dfrac{\ell}{\beta}, & \ell \gg \beta \qquad \text{classical extensive} \\ \\ \dfrac{c}{3} \log \dfrac{\ell}{a}, & \ell \ll \beta \qquad T = 0 \text{ non} - \text{extensive} \end{cases}$$

2. Finite size: orient the branch cut perpendicular to the axis $\beta \to L$ and $w \to iw$

$$S_A \sim \frac{c}{3} \log \left(\frac{L}{\pi a} \sin \frac{\pi \ell}{L} \right) + c_1'$$

It is symmetric under $\ell \to L - \ell$. It is maximal when $\ell = L/2$

3. Open boundaries: semi-infinite system



If $L = \infty$ and $T = 0$, it is uniformised by $z = \left(\frac{w-i\ell}{w+i\ell}\right)^{1/n}$

$$\mathrm{Tr}\,\rho_A^n \simeq \tilde{c}_n \left(\frac{2\ell}{a}\right)^{(c/12)(n-1/n)} \qquad \Rightarrow S_A \simeq \frac{c}{6}\log\frac{2\ell}{a} + \tilde{c}_1'$$

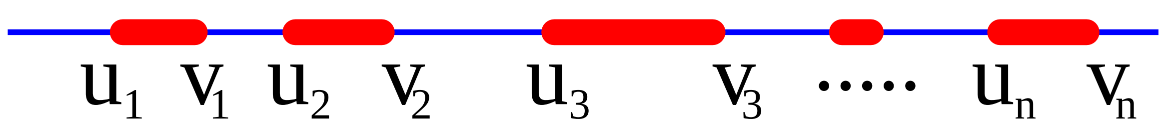
and at finite temperature β^{-1} and finite size

$$S_A(\beta) \simeq \frac{c}{6}\log\left(\frac{\beta}{\pi a}\sinh\frac{2\pi\ell}{\beta}\right) + \tilde{c}_1'$$

$$S_A(L) \simeq \frac{c}{6}\log\left(\frac{2L}{\pi a}\sin\frac{\pi\ell}{L}\right) + \tilde{c}_1'$$

Note: $\tilde{c}_1' - c_1' = g$ boundary entropy [Affleck, Ludwig]

4. General case:



Uniformised by $z = \prod_i (w - w_i)^{\alpha_i}$, with $\sum_i \alpha_i = 0$ ($w_k = u_i$ or v_j)

$$S_A = \frac{c}{3}\left(\sum_{j\le k}\log\frac{v_k - u_j}{a} - \sum_{j<k}\log\frac{u_k - u_j}{a} - \sum_{j<k}\log\frac{v_k - v_j}{a}\right) + Nc_1'$$

A similar expression holds in the case of a boundary, with half of the w_i corresponding to the image points

Entropy in non critical systems

Question: What about the entanglement entropy in the so-called critical domain, where $g \neq g_c$, but $|g-g_c| \ll 1$, i.e. the correlation length $\xi = |g-g_c|^{-\nu}$ is large but finite?

Following the line of the c-theorem proof, we showed

$$\boxed{S_A = \mathcal{A}\frac{c}{6}\log\frac{\xi}{a}}$$

where $\mathcal{A}$ is the number of boundary points between A and B (1D area).

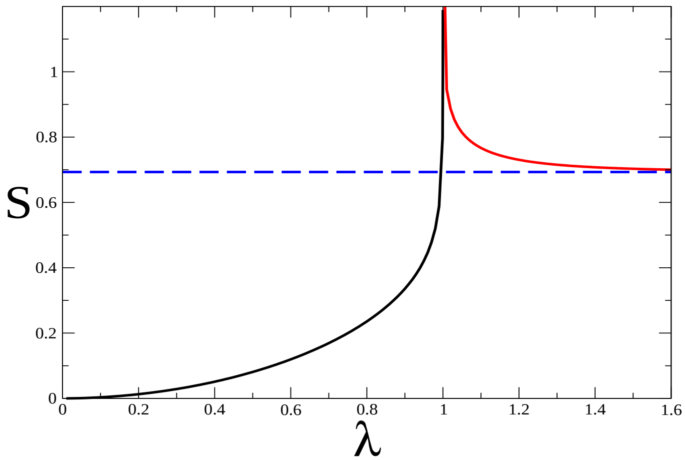
We checked this result in some cases with $\mathcal{A} = 1$

- Gaussian Massive FT
- Ising model in a transverse magnetic field

$$H_I = -\sum_{n=1}^{L-1}\sigma_n^x - \lambda\sum_{n=1}^{L-1}\sigma_n^z\sigma_{n+1}^z$$

by means of the corner transfer matrix ($\epsilon = \epsilon(\lambda)$)

$$S_A = \begin{cases} \epsilon\displaystyle\sum_{j=0}^{\infty}\frac{2j}{1+e^{2j\epsilon}} + \sum_{j=0}^{\infty}\log(1+e^{-2j\epsilon})\,, & \lambda > 1 \\ \epsilon\displaystyle\sum_{j=0}^{\infty}\frac{2j+1}{1+e^{(2j+1)\epsilon}} + \sum_{j=0}^{\infty}\log(1+e^{-(2j+1)\epsilon})\,, & \lambda < 1 \end{cases}$$



For $\lambda \to 1$

$$S \simeq \frac{1}{12}\log\xi + C_1$$

- XXZ model, similar results but $c = 1$
- In the finite slit geometry (i.e. $\mathcal{A} = 2$), it was exactly calculated by Its et al. and Peschel for the XY chain, finding agreement!

Dynamics of Entanglement

How entanglement evolves when the system is prepared in a state that is *not* an eigenstate?

EG: Ising model in a transverse field with $H(h)$:

- Prepare the system in a pure state $|h_0\rangle$ (ground state of $H(h_0)$)
- Let it evolve according to $H(h)$ with $h \neq h_0$ (ie at $t = 0$ the field has been quenched)

$$\Downarrow$$

$$|\psi(t)\rangle = e^{-iH(h)t}|h_0\rangle$$

$$\Downarrow$$

$$\rho_A(t) = \mathrm{Tr}_B\, e^{-iH(h)t}|\psi_0\rangle\langle\psi_0|e^{iH(h)t}$$

Tr_B and H do not "commute" $\Rightarrow$ non trivial evolution

Note: the system does not relax to the ground state

How can we study this problem with QFT?

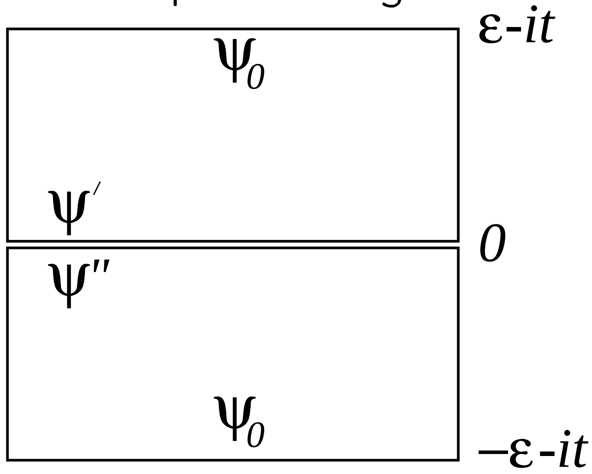
Matrix elements of the density matrix at time t

$\langle\psi''(x'')|\rho(t)|\psi'(x')\rangle = Z_1^{-1}\langle\psi''(x'')|e^{-itH-\epsilon H}|\psi_0(x)\rangle\langle\psi_0(x)|e^{+itH-\epsilon H}|\psi'(x')\rangle$

We use $e^{-\epsilon H}$ to make the path integral convergent!

Important: We'll see at the end if it is justified to remove ϵ

Each of the factors may be represented by an analytically continued path integral in imaginary time:



CFT results

In CFT the calculation is done in imaginary times τ_1 and τ_2, and then it is analytically continued to $\tau_1 = \epsilon - it$, $\tau_2 = -\epsilon - it$

The strip geometry is obtained by transforming the upper half-plane with $w = (2\epsilon/\pi)\log z$

In the upper half-plane with boundary

$$\mathrm{Tr}\,\rho_A^n = \langle\Phi_n\Phi_{-n}\rangle \sim c_n \left(\frac{|z_1-\bar{z}_2||z_2-\bar{z}_1|}{|z_1-z_2||\bar{z}_1-\bar{z}_2||z_1-\bar{z}_1||z_2-\bar{z}_2|}\right)^{2n\Delta_n}$$

$z_1 = \rho^{-1}e^{i\pi\tau_1/2\epsilon} = z_2^{-1}$ where $\rho = e^{\pi\ell/4\epsilon}$

Algebra . . . ℓ/ϵ and $t/\epsilon \gg 1$ . . .

$$c_n(\pi/2\epsilon)^{4n\Delta_n}\left(\frac{e^{\pi\ell/2\epsilon}+e^{\pi t/\epsilon}}{e^{\pi\ell/2\epsilon}\cdot e^{\pi t/\epsilon}}\right)^{2n\Delta_n}$$

Differentiating wrt n

$$S_A(t) \sim \begin{cases} \dfrac{\pi c t}{6\epsilon} & (t < \ell/2)\,, \\ \dfrac{\pi c\,\ell}{12\epsilon} & (t > \ell/2)\,, \end{cases}$$

$S_A(t)$ increases linearly until it saturates at $t = \ell/2$.

ϵ enters in an essential way: in a continuum FT a state like $|\psi_0\rangle$ has infinitely large mean energy

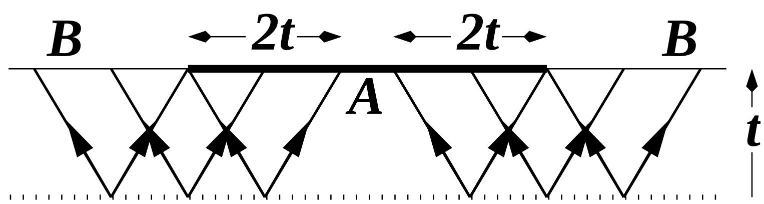
Physical Interpretation

$|\psi_0\rangle$ has a very high energy relative to the ground state

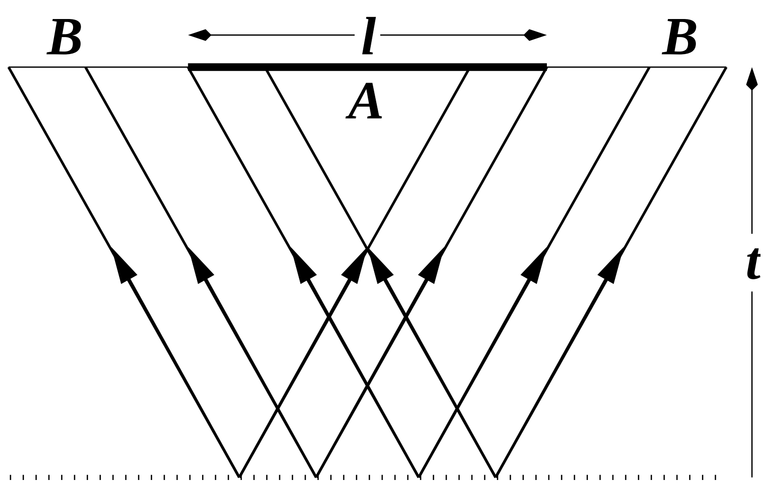
$\Rightarrow$ acts as a source of "particles" propagating at the speed of light

Particles emitted from different points are incoherent, but pairs of particles moving to the left or right from a given point are highly entangled

The field at some point x in A will be entangled with that at a point $x' \in B$ if a left (right) moving particle arriving at x is entangled with a right (left) moving particle arriving at x', and this can happen only if $x \pm t \sim x' \mp t$



2t < l



2t > l

$S_\ell(t)$ is proportional to the length of the interval in x for which this is true, reproducing the CFT result

Generalizable to the case when A consists of several disjoint intervals. $S_A(t)$ is not always non-decreasing:

EG, $A =$ regular array of intervals
$\Rightarrow S_\ell$ oscillates in a saw-tooth fashion

Lattice calculation

We considered the transverse Ising chain

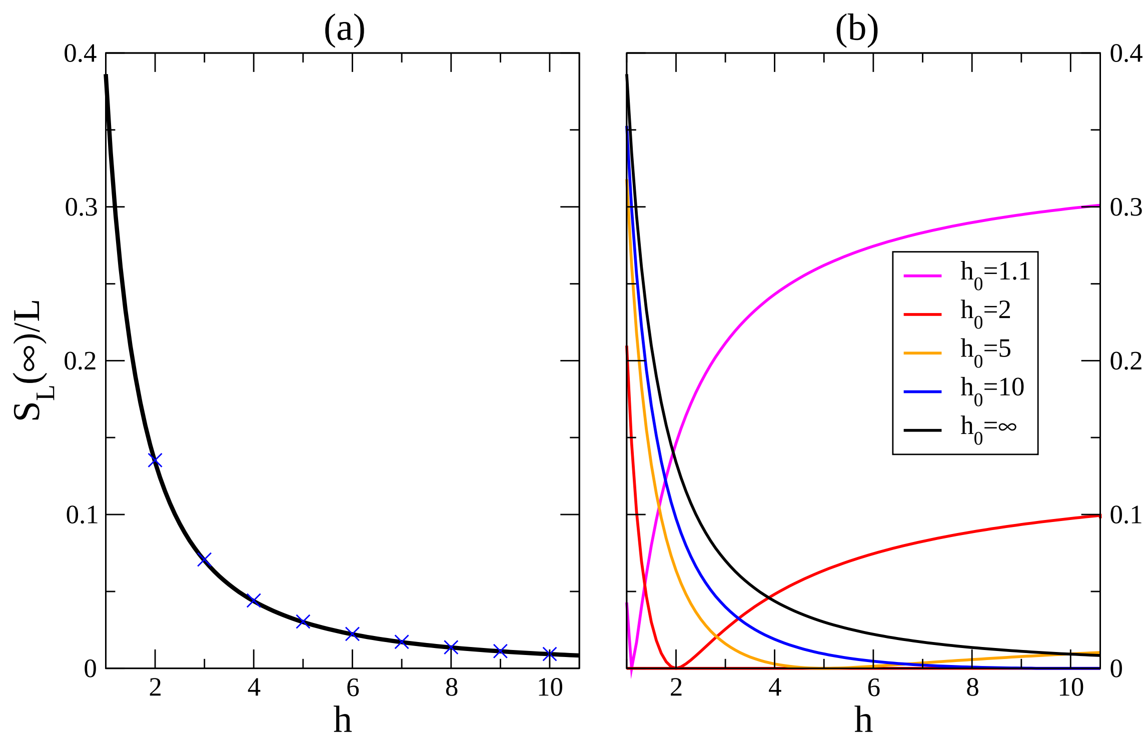
$$H_I(h) = -\frac{1}{2}\sum_j [\sigma_j^x \sigma_{j+1}^x + h\sigma_j^z]$$

$t \to \infty$, Analytic calculations

...very cumbersome ...

$$S_\ell = \frac{\ell}{2\pi}\int_0^{2\pi} d\varphi H\left(\frac{1-\cos\varphi(h+h_0)+hh_0}{\sqrt{(h^2+1-2h\cos\varphi)(h_0^2+1-2h_0\cos\varphi)}}\right)$$

with $H(x) = -\frac{1+x}{2}\log\frac{1+x}{2} - \frac{1-x}{2}\log\frac{1-x}{2}$

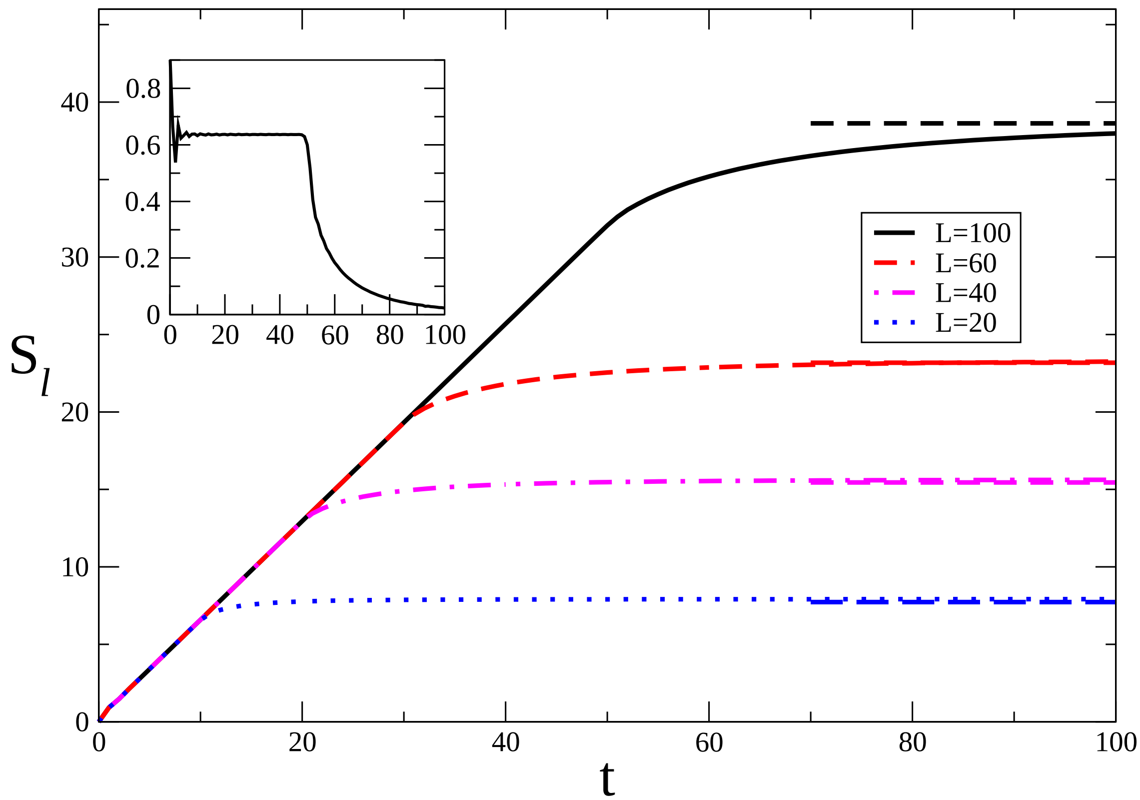


always linear in ℓ, not only at the critical point

Curiosity: $S_\ell(\infty)$ is symmetric under the exchange of h and h_0 (??)

Finite time: numerical calculations

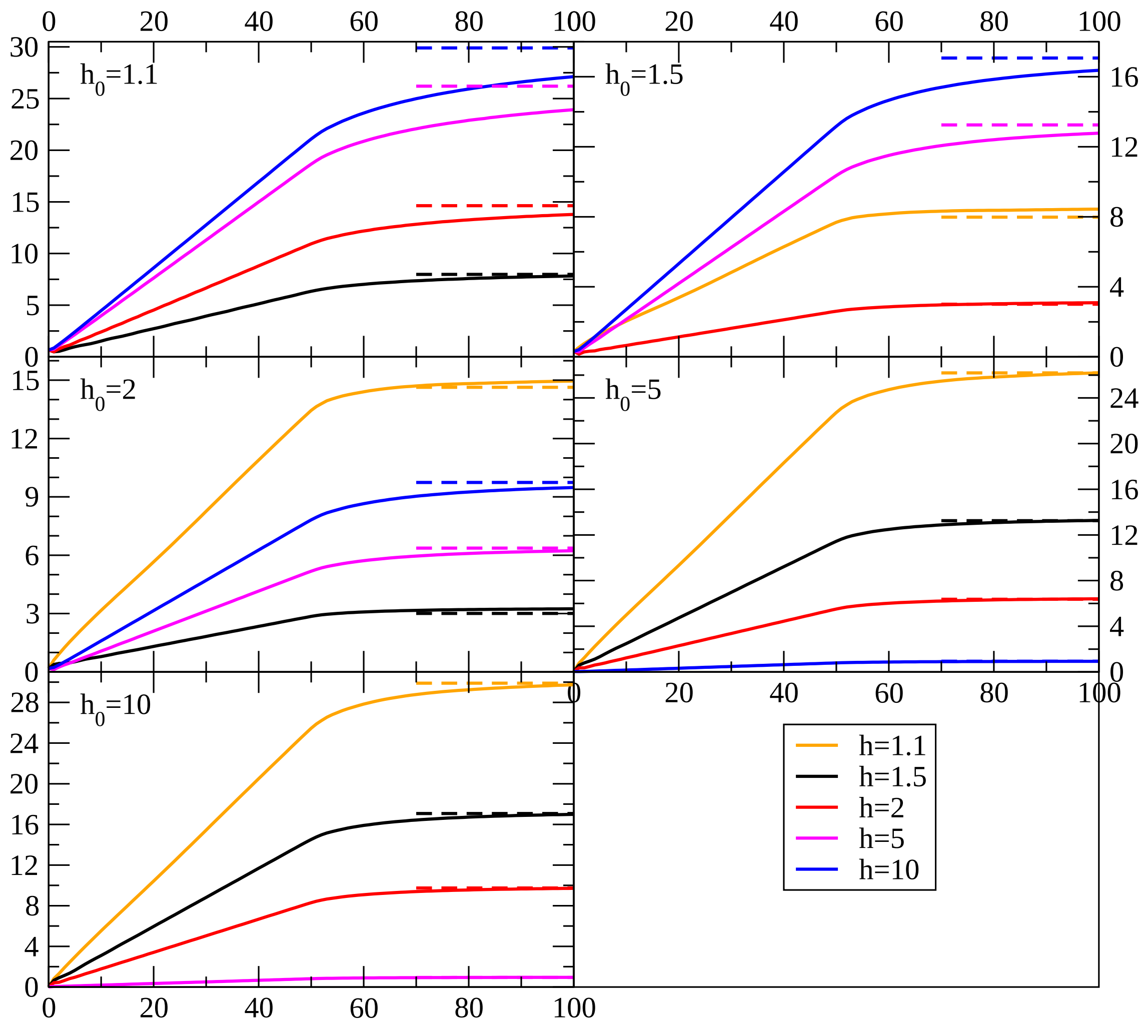
$h_0 = \infty$, $h = 1$:



Linear for $t < \ell/2$!!

But does not saturate at $\ell/2$ (??)

General h, h_0:



- Crossover at $t^* = \ell/2$!!

- Different approach to the asymptotic value for h and h_0 interchanged

Discussions

$S_\ell(t)$ increases linearly with time up to $t^* = \ell/2$, but (as a difference with CFT and the previous argument)

$$R \equiv \frac{(\partial S_A/\partial t)_{t<t^*}}{2(\partial S_A/\partial \ell)_{t\gg t^*}} \neq 1$$

How we can match these (apparently) different results?

There are other excitations traveling with speed $v < 1$

Suppose that the rate of production of pair of particles of momenta (p', p'') is $f(p', p'')$

with dispersion relation $E = E(p) \Rightarrow v_p = dE/dp \leq 1$

$$S_A(t) \approx \int_{x'\in A} dx' \int_{x''\in B} dx'' \int_{-\infty}^{\infty} dx \int f(p', p'') dp' dp'' \delta\left(x' - x - v_{p'} t\right) \delta\left(x'' - x - v_{p''} t\right)$$

When A is the interval of length ℓ

$$S_A(t) \propto t \int_{-\infty}^{0} dp' \int_{\ell-(v_{-p'}+v_{p''})t>0} dp'' f(p', p'')(v_{-p'} + v_{p''}) + \ell \int_{-\infty}^{0} dp' \int_{\ell-(v_{-p'}+v_{p''})t<0} dp'' f(p', p'')$$

$|v_p| \leq 1 \Rightarrow 2^{\text{nd}}$ term is zero if $t < \ell/2 \Rightarrow S_A(t) \propto t$

For $t \to \infty$, the first term is negligible $\Rightarrow S_A \propto \ell$

Unless $|v| = 1$ everywhere (CFT) $S_A \not\propto \ell$ for all $t > t^*$

$$R = \frac{\int_{-\infty}^{0} dp' \int_0^{\infty} dp'' f(p', p'')[v_{-p'} + v_{p''}]}{2\int_{-\infty}^{0} dp' \int_0^{\infty} dp'' f(p', p'')} \leq 1$$

The correction term is a power law $S_\ell \propto \ell(1 - (\ell/t)^\alpha)$

Future directions

Consider $|\psi_0\rangle$ having a finite energy above the ground state. Will it relax to $c/3 \log \ell$? How?