Generalised permutation branes

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To appear soon...
The issue of classifying branes

Major steps in the classification of branes:
- Untwisted branes (Cardy’s construction)
- Twisted branes
- [ (Super)conformal branes (only for very special models)]
- Symmetry breaking branes

⇒ What else?
⇒ Need to look for branes with special properties

Proposal: Employ predictions from K-theory
- K-groups for group manifolds are known explicitly
- Charge carrying branes should have an enhanced symmetry
- One can get some hints about the geometry

Prototype: $SU(2) \times SU(2)$ WZW model
⇒ Full control for equal levels, but otherwise?
A short introduction to WZW models

- WZW models: 2d non-linear $\sigma$-models on a Lie group $G$

- Action: Based on non-degenerate invariant form $\langle \cdot, \cdot \rangle$,

$$S[g] \sim \int_{\Sigma_2} \mathcal{L}_{\text{kin}} + \int_{B_3} \omega^{\text{WZ}} - \int_{D_2} \omega_2$$

- WZ-form: $3\omega^{\text{WZ}} = \langle g^{-1} dg, [g^{-1} dg, g^{-1} dg] \rangle$, $d\omega_2 = \omega^{\text{WZ}}|_{D_2}$

- For $G$ simple: $\langle \cdot, \cdot \rangle = k \kappa(\cdot, \cdot)$

- $\hat{G}_k \times \hat{G}_k$ loop group symmetry $g \mapsto g_L(z)gg_R(\bar{z})^{-1}$
Maximally symmetric branes on group manifolds

- Branes wrap twisted conjugacy classes

\[ \mathcal{D} = \mathcal{C}_f(\Omega) = \{ h f \Omega(h^{-1}) | h \in G \} \]

- Diagonal \( \hat{G}_k \) loop group symmetry via \( g \mapsto h g \Omega(h^{-1}) \)

- \( \Omega \) must be \textit{isometric}

- \( f \) (and hence \( \omega_2 \)) is quantised

- An example: \( SU(2) \cong S^3 \)
  - Branes are labelled by integers \( \lambda = 0, \ldots, k \)
  - \( \lambda = 0, k \) correspond to 0-branes, the rest to \( S^2 \)-branes
Boundary RG flow invariants and K-theory

- Basic ideas:
  - Space-time physics (tachyon condensation) $\leftrightarrow$ RG flows
  - Conserved charges $\leftrightarrow$ K-theory: $nV \oplus m\bar{V} \cong (n - m)V$
  - Non-trivial H-flux $\Rightarrow$ need to use twisted K-theory

- An example: $SU(2)$
  - Every brane can be obtained from 0-branes
  - Charge group $K^\tau(SU(2)_k) = \mathbb{Z}_{k+2}$
For a simple group $G$ one has

$$K^\tau(G_k) = (\Z_d)^{2^{r-1}}$$

where $r = \text{rank } G$ and $d$ is determined by $G$ and $k$.

A generalisation of the Küneth formula implies

$$K^\tau(G_{k_1} \times G_{k_2}) = (\Z_{\gcd(d_1,d_2)})^{2^{2r-1}}$$

For $SU(2) \times SU(2)$ one thus obtains

$$K^\tau(SU(2)_{k_1} \times SU(2)_{k_2}) = 2 \cdot \Z_{\gcd(k_1+2,k_2+2)}$$

**Question:** Which branes correspond to the “new” charges?
Product groups: The case of equal levels

Consider the product group $G \times G$ with metric

$$\langle \cdot, \cdot \rangle = k (\kappa_1 (\cdot, \cdot) + \kappa_2 (\cdot, \cdot))$$

What kind of automorphisms do we have?

$$\Omega = \Omega_1 \times \Omega_2 \Rightarrow \text{factorising branes}$$

$$D = C_{f_1} (\Omega_1) \times C_{f_2} (\Omega_2)$$

Exchange automorphism $\Omega(g_1, g_2) = (g_2, g_1)$

$$D = C_f (\Omega) = \{(g_1 f g_2^{-1}, g_2 f g_1^{-1}) \mid g_1, g_2 \in G\}$$

The simplest permutation brane is given by $f = 1$,

$$D = \{(g, g^{-1}) \mid g \in G\}$$
Product groups: The case of different levels

Now consider the same group $G \times G$ with metric

$$\langle \cdot, \cdot \rangle = k_1 \kappa_1(\cdot, \cdot) + k_2 \kappa_2(\cdot, \cdot)$$

What kind of automorphisms do we have now?

- $\Omega = \Omega_1 \times \Omega_2 \Rightarrow$ factorising branes

$$\mathcal{D} = \mathcal{C}_f(\Omega_1) \times \mathcal{C}_f(\Omega_2)$$

The exchange automorphism $\Omega(g_1, g_2) = (g_2, g_1)$ still exists but it is not isometric anymore!

Proposal: The simplest permutation brane is deformed to

$$\mathcal{D} = \{ (g^{k_2'}, g^{-k_1'}) \mid g \in G \} \quad \text{with} \quad k_i' = k_i / \gcd(k_1, k_2)$$

The symmetry is broken to $\hat{G}_{k_1+k_2}$
Lagrangian approach

- Trivialising two-form for the simplest permutation brane

\[ \omega_2 = k_1 \sum_{j=1}^{k_2'-1} (k_2' - j) \text{tr} [\text{Ad}_{g^j} (g^{-1}dg)g^{-1}dg] + (1 \leftrightarrow 2) \]

- One can prove gluing conditions \( J_1 + J_2 = \bar{J}_1 + \bar{J}_2 \)

- Concrete formulas for \( SU(2) \times SU(2) \):

  - \( g = \begin{pmatrix} \cos \psi + i \cos \theta \sin \psi & \sin \psi \sin \theta e^{i\phi} \\ -\sin \psi \sin \theta e^{-i\phi} & \cos \psi - i \cos \theta \sin \psi \end{pmatrix} \Rightarrow g(\psi)^n = g(n\psi) \)
  
  - \( ds^2 = \sum k_i (d\psi_i^2 + \sin^2 \psi (d\theta_i^2 + \sin^2 \theta_i d\phi_i)) \)
  
  - \( B = \sum k_i (\psi_i - \frac{1}{2} \sin(2\psi_i)) \sin \theta_i d\theta_i \wedge d\phi_i, \quad H = dB \)

- Brane embedding: \( \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} k_2' \psi \\ -k_1' \psi \end{pmatrix}, \quad \theta_i = \theta, \phi_i = \phi \)

- \( \omega_2 \sim [k_2 \sin(2k_1' \psi) - k_1 \sin(2k_2' \psi)] \sin \theta \, d\theta \wedge d\phi \)
Dirac-Born-Infeld approach

- Branes minimise the DBI action

\[ S_D = \int e^{-\Phi} \sqrt{\det(\hat{g} + \omega_2)} + \cdots \]

- Brane embedding: \( X^\mu = X^\mu(Y^i) \)
- Induced metric \( \hat{g}_{ij} = g_{\mu\nu} \partial_i X^\mu \partial_j X^\nu \)

- For \( \Phi = 0 \) the EOM read

\[ \text{tr} \left[ (\hat{g} + \omega_2)^{-1} \Omega^\mu \right] = 0 \]

\[ \Omega_{ij}^\mu = \partial_i \partial_j X^\mu + \Gamma_{i\nu\rho}^\mu \partial_i X^\nu \partial_j X^\rho - \hat{\Gamma}_{ij}^k \partial_k X^\mu \]

- Generalised connection \( \Gamma = \Gamma_{LC}(g) - \frac{1}{2} H \)

- Works out for the simplest permutation brane on \( G \times G \)!
Non-trivial defect lines from folding

- Folding maps defect lines between CFT$_1$ and CFT$_2$ to boundaries in CFT$_1 \times$ CFT$_2$

\[ \begin{array}{ccc}
\text{G}_{k_1} & \text{G}_{k_2} \\
\end{array} \rightarrow \begin{array}{c}
\text{G}_{k_1} \times \text{G}_{k_2}
\end{array} \]

- Our construction gives rise to new non-trivial defects between two WZW models based on the same group but at different levels.

**Reminder:** The old construction used the decomposition

\[ \frac{G_{k_1} \times G_{k_2}}{G_{k_1+k_2}} \times G_{k_1+k_2} \rightarrow G_{k_1} \times G_{k_2} \]
Generalisation to cosets

How to extend our construction to cosets \( G/H \times G/H \)?

The group \( G \) and the coset \( G/T \) (\( T = \text{maximal torus} \)) are related by a marginal bulk deformation.

Natural guess for the brane geometry:

\[
\mathcal{D} = \left\{ \left( (gh)^{k_2'}, (hg)^{-k_1'} \right) \mid g \in G, h \in H \right\} \subset G \times G
\]

But: Open whether DBI is satisfied for this proposal

Aim: New branes in Calabi-Yau manifolds (Gepner models)

\[
\frac{SU(2)^{k_1} \times U(1)^2}{U(1)^{k_1+2}} \times \ldots \times \frac{SU(2)^{k_n} \times U(1)^2}{U(1)^{k_n+2}}
\]
Towards a CFT description?

Denote by $\text{MM}_p$ the $p^{th}$ minimal model

Consider the following CFT

$$\text{MM}_p \times \text{MM}_p \text{ with a permutation brane}$$

Then one should be able to follow the combined bulk/boundary flow to the CFT $(q < p)$

$$\text{MM}_q \times \text{MM}_p \text{ with some brane configuration } X$$

Questions:

- How does $X$ look like?
- Is it an elementary brane or composite?
- Can one identify the symmetry preserved?
Outlook

Unfortunately not enough time to discuss:

Higher dimensional branes ($f \neq 1$)

Arbitrary permutations in $G \times \cdots \times G$

Open questions:

- DBI calculations for all these branes
- Check of the EOM
- Calculation of energies
- Identification of the precise symmetry preserved
- Boundary states
- Calculation of boundary entropies ($g$-factors)
- Comparison with DBI results
- Application to Gepner models
- Analogous construction for principal chiral models?