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Non-Hermitian representations for noncommutative spaces

Andreas Fring

Pseudo-Hermitian Hamiltonians in Quantum Physics XII
Koç University, Istanbul,02-06 July, 2013

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S. Dey, AF; L. Gouba; J. Phys. A45 (2012) 385302,
S. Dey, AF; Phys. Rev. D86 (2012) 064038,
S. Dey, AF; L. Gouba, P. Castro; Phys. Rev. D87 (2013) 084033,
S. Dey, AF; B. Khantoul, arXiv:1302.4571

Noncommutative spaces

- Flat (abelian) noncommutative space

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}$$

In 3D:

$$\begin{aligned} &[x_0, y_0] = i\theta_1, \quad &[x_0, z_0] = i\theta_2, \quad &[y_0, z_0] = i\theta_3, \\ &[x_0, p_{x_0}] = i\hbar, \quad &[y_0, p_{y_0}] = i\hbar, \quad &[z_0, p_{z_0}] = i\hbar, \end{aligned} \qquad \theta_1, \theta_2, \theta_3 \in \mathbb{R}$$

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- Snyder spaces, from twists:

$$[x_i, x_j] = i\theta(x_i p_j - x_j p_i)$$

- Minimal uncertainty relations, from q-deformed algebras?:

$$[x_i, x_j] \approx i\theta(x_j)^2$$

Minimal lengths, areas and volumes

Uncertainty relation:

$$\Delta A \Delta B \geq \frac{1}{2} \left| \langle [A, B] \rangle_{\rho} \right|$$

- Standard case:
  $[A, B] = const$; give up knowledge about $B \Rightarrow \Delta A = 0$

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- Noncommutative case:
  $[A, B] \approx B^2$; even give up knowledge about $B \Rightarrow \Delta A \neq 0$
- For instance:

$$[X, P] = i\hbar \left(1 + \tau P^2\right)$$

$\Rightarrow$ minimal length

$$\Delta X_{\min} = \hbar\sqrt{\tau}\sqrt{1 + \tau \left\langle P^2 \right\rangle_\rho}$$

from minimizing with $(\Delta A)^2 = \langle A^2 \rangle - \langle A \rangle^2$

Oscillator algebra vs canonical commutators

Deformed oscillator algebra:

$$A_i A_j^\dagger - q^{2\delta_{ij}} A_j^\dagger A_i = \delta_{ij}, \; [A_i^\dagger, A_j^\dagger] = [A_i, A_j] = 0, i, j = 1, 2, 3; q \in \mathbb{R}$$

The limit $q \to 1$ gives standard Fock space $A_i \to a_i$:

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q-deformed Fock space representation (1D):

$$|n\rangle_q := \frac{(A^\dagger)^n}{\sqrt{[n]_q!}} |0\rangle, \quad [n]_q := \frac{1 - q^{2n}}{1 - q^2}, \quad [n]_q! := \prod_{k=1}^{n} [k]_q$$

$$A|0\rangle = 0, \quad \langle 0|0\rangle = 1,$$

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[M. Arik, D.D. Coon; J. Math. Phys. 17 (1976) 524
A.J. Macfarlane, J. Phys. A22 (1989) 4581
P.P. Kulish, E.V. Damaskinsky, J. Phys. A23 (1990) L415]

What is the relation to the observables?

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- linear Ansatz for 3D flat noncommutative space:

$$\varphi_i = \sum_{j=1}^{3} \kappa_{ij} \; a_j + \lambda_{ij} a_j^\dagger, \qquad \text{for } \vec{\varphi} = \{x_0, y_0, z_0, p_{x_0}, p_{y_0}, p_{z_0}\}$$

$\Rightarrow$ 72 free parameters

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- For $A_i, A_i^\dagger \to X,\ Y,\ Z,\ P_x,\ P_y,\ P_z$:
  - we do not even know the commutation relations
  - representations turn out to be non-Hermitian

$\mathcal{PT}$-symmetric noncommutative spaces

P. Giri, P. Roy, Eur. Phys. C60 (2009) 157: $\nexists$ $\mathcal{PT}$-symmetry

$$[x_0, y_0] = i\theta_1, \qquad [x_0, z_0] = i\theta_2, \qquad [y_0, z_0] = i\theta_3, \qquad \theta_1, \theta_2, \theta_3 \in \mathbb{R}$$
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$$\mathcal{PT}_{\pm}: \quad x_0 \to \pm x_0, \quad y_0 \to \mp y_0, \quad z_0 \to \pm z_0, \quad i \to -i,$$
$$p_{x_0} \to \mp p_{x_0}, \quad p_{y_0} \to \pm p_{y_0}, \quad p_{z_0} \to \mp p_{z_0}, \quad \theta_2 = 0.$$

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- Reduce number of free parameters.

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- Reduce number of free parameters.
- Models on these spaces will have the usual nice properties.

Oscillator algebras of flat noncommutative spaces

$\mathcal{PT}_{\pm}$-symmetric Ansatz:

$$\begin{aligned} a_1 &= \alpha_1 x_0 + i\alpha_2 y_0 + \alpha_3 z_0 + i\alpha_4 p_{x_0} + \alpha_5 p_{y_0} + i\alpha_6 p_{z_0}, \\ a_2 &= \alpha_7 x_0 + i\alpha_8 y_0 + \alpha_9 z_0 + i\alpha_{10} p_{x_0} + \alpha_{11} p_{y_0} + i\alpha_{12} p_{z_0}, \\ a_3 &= \alpha_{13} x_0 + i\alpha_{14} y_0 + \alpha_{15} z_0 + i\alpha_{16} p_{x_0} + \alpha_{17} p_{y_0} + i\alpha_{18} p_{z_0}, \end{aligned}$$

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\end{aligned}
$$

Solution:

$$
\begin{aligned}
p_{x_0} &= \frac{\alpha_{12}\alpha_{14} - \alpha_8\alpha_{18}}{2i \det M_2} a_1^- + \frac{\alpha_2\alpha_{18} - \alpha_6\alpha_{14}}{2i \det M_2} a_2^- + \frac{\alpha_6\alpha_8 - \alpha_2\alpha_{12}}{2i \det M_2} a_3^-, \\
p_{y_0} &= \frac{\alpha_7\alpha_{15} - \alpha_9\alpha_{13}}{2 \det M_1} a_1^+ + \frac{\alpha_3\alpha_{13} - \alpha_1\alpha_{15}}{2 \det M_1} a_2^+ + \frac{\alpha_1\alpha_9 - \alpha_3\alpha_7}{2 \det M_1} a_3^+, \\
p_{z_0} &= \frac{\alpha_8\alpha_{16} - \alpha_{10}\alpha_{14}}{2i \det M_2} a_1^- + \frac{\alpha_4\alpha_{14} - \alpha_2\alpha_{16}}{2i \det M_2} a_2^- + \frac{\alpha_2\alpha_{10} - \alpha_4\alpha_8}{2i \det M_2} a_3^-,
\end{aligned}
$$

with

$$
\begin{aligned}
a_i^{\pm} &= a_i \pm a_i^{\dagger} \\
(M_l)_{jk} &= 6j + 2k + l - 8 \qquad \text{for} \qquad l = 1, 2.
\end{aligned}
$$

Noncommutative spaces from oscillator algebras

Similarly $\mathcal{PT}_{\pm}$-symmetric Ansatz:

$$
\begin{aligned}
X &= \hat{\kappa}_1(A_1^\dagger + A_1) + \hat{\kappa}_2(A_2^\dagger + A_2) + \hat{\kappa}_3(A_3^\dagger + A_3), \\
Y &= i\hat{\kappa}_4(A_1^\dagger - A_1) + i\hat{\kappa}_5(A_2^\dagger - A_2) + i\hat{\kappa}_6(A_3^\dagger - A_3), \\
Z &= \hat{\kappa}_7(A_1^\dagger + A_1) + \hat{\kappa}_8(A_2^\dagger + A_2) + \hat{\kappa}_9(A_3^\dagger + A_3), \\
P_x &= i\check{\kappa}_{10}(A_1^\dagger - A_1) + i\check{\kappa}_{11}(A_2^\dagger - A_2) + i\check{\kappa}_{12}(A_3^\dagger - A_3), \\
P_y &= \check{\kappa}_{13}(A_1^\dagger + A_1) + \check{\kappa}_{14}(A_2^\dagger + A_2) + \check{\kappa}_{15}(A_3^\dagger + A_3), \\
P_z &= i\check{\kappa}_{16}(A_1^\dagger - A_1) + i\check{\kappa}_{17}(A_2^\dagger - A_2) + i\check{\kappa}_{18}(A_3^\dagger - A_3),
\end{aligned}
$$

with $\hat{\kappa}_i = \kappa_i\sqrt{\hbar/(m\omega)}$ for $i = 1, \ldots, 9$
$\check{\kappa}_i = \kappa_i\sqrt{m\omega\hbar}$ for $i = 10, \ldots, 18$

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$\check{\kappa}_i = \kappa_i\sqrt{m\omega\hbar}$ for $i = 10, \ldots, 18$

Note: X, Y, Z, P_x, P_y, P_z are non-Hermitian in the usual space

Compute non-vanishing commutators:

$$
\begin{aligned}
[X, Y] &= 2i \sum_{j=1}^{3} \hat{\kappa}_j \hat{\kappa}_{3+j} \left[1 + \left(q^2 - 1\right)\right] A_j^\dagger A_j, \\
[Y, Z] &= -2i \sum_{j=1}^{3} \hat{\kappa}_{3+j} \hat{\kappa}_{6+j} \left[1 + \left(q^2 - 1\right)\right] A_j^\dagger A_j, \\
[X, P_x] &= 2i \sum_{j=1}^{3} \hat{\kappa}_j \check{\kappa}_{9+j} \left[1 + \left(q^2 - 1\right)\right] A_j^\dagger A_j, \\
[Y, P_y] &= -2i \sum_{j=1}^{3} \hat{\kappa}_{3+j} \check{\kappa}_{12+j} \left[1 + \left(q^2 - 1\right)\right] A_j^\dagger A_j, \\
[Z, P_z] &= 2i \sum_{j=1}^{3} \hat{\kappa}_{6+j} \check{\kappa}_{15+j} \left[1 + \left(q^2 - 1\right)\right] A_j^\dagger A_j, \\
[P_x, P_y] &= -2i \sum_{j=1}^{3} \check{\kappa}_{9+j} \check{\kappa}_{12+j} \left[1 + \left(q^2 - 1\right)\right] A_j^\dagger A_j, \\
[P_y, P_z] &= 2i \sum_{j=1}^{3} \check{\kappa}_{12+j} \check{\kappa}_{15+j} \left[1 + \left(q^2 - 1\right)\right] A_j^\dagger A_j, \\
[X, P_z] &= 2i \sum_{j=1}^{3} \hat{\kappa}_j \check{\kappa}_{15+j} \left[1 + \left(q^2 - 1\right)\right] A_j^\dagger A_j, \\
[Z, P_x] &= 2i \sum_{j=1}^{3} \hat{\kappa}_{6+j} \check{\kappa}_{9+j} \left[1 + \left(q^2 - 1\right)\right] A_j^\dagger A_j,
\end{aligned}
$$

A particular $\mathcal{PT}_{\pm}$-symmetric solution

$$\kappa_1 = \kappa_4 = \kappa_5 = \kappa_8 = \kappa_{10} = \kappa_{12} = \kappa_{13} = \kappa_{14} = \kappa_{17} = \kappa_{18} = 0$$

$$[X, Y] = i\theta_1 + i\frac{q^2-1}{q^2+1}\frac{\theta_1}{\hbar}\left[\frac{m\omega}{2\kappa_6^2}Y^2 + \frac{2\kappa_6^2}{m\omega}P_y^2\right]$$

$$[Y, Z] = i\theta_3 + i\frac{q^2-1}{q^2+1}\frac{\theta_3}{\hbar}\left[\frac{m\omega}{2\kappa_6^2}Y^2 + \frac{2\kappa_6^2}{m\omega}P_y^2\right]$$

$$[X, P_x] = i\hbar + i\frac{q^2-1}{q^2+1}2m\omega\left[\kappa_{11}^2 X^2 + \frac{P_x^2/4}{m^2\omega^2\kappa_{11}^2} + \frac{\theta_1^2\kappa_{11}^2 P_y^2}{\hbar^2} + \frac{\theta_1\kappa_{11}^2 X P_y}{\hbar/2}\right]$$

$$[Y, P_y] = i\hbar + i\frac{q^2-1}{q^2+1}2m\omega\left[\frac{1}{4\kappa_6^2}Y^2 + \frac{\kappa_6^2}{m^2\omega^2}P_y^2\right]$$

$$[Z, P_z] = i\hbar + i\frac{q^2-1}{q^2+1}2m\omega\left[\frac{Z^2}{4\kappa_7^2} + \frac{\kappa_7^2}{m^2\omega^2}P_z^2 + \frac{\theta_3^2}{4\hbar^2\kappa_7^2}P_y^2 - \frac{\theta_3 Z P_y}{2\hbar^2\kappa_7^2}\right]$$

with constraints

$$\hat{\kappa}_2 = \frac{\hbar}{2\check{\kappa}_{11}}, \quad \hat{\kappa}_3 = \frac{\theta_1}{2\hat{\kappa}_6}, \quad \hat{\kappa}_9 = -\frac{\theta_3}{2\hat{\kappa}_6}, \quad \check{\kappa}_{15} = -\frac{\hbar}{2\hat{\kappa}_6}, \quad \check{\kappa}_{16} = \frac{\hbar}{2\hat{\kappa}_7}$$

Reduced three dimensional solution for $q \to 1$

- Set $\check{\kappa}_{11} = m\omega\hat{\kappa}_6$, $\kappa_7 = 1/2\kappa_6$, $q = \exp(2\tau\kappa_6^2)$, then $\kappa_6 \to 0$:

$$
\begin{aligned}
[X, Y] &= i\theta_1 \left(1 + \hat{\tau} Y^2\right), \quad [Y, Z] = i\theta_3 \left(1 + \hat{\tau} Y^2\right), \\
[X, P_x] &= i\hbar \left(1 + \check{\tau} P_x^2\right), \quad [Y, P_y] = i\hbar \left(1 + \hat{\tau} Y^2\right) \\
[Z, P_z] &= i\hbar \left(1 + \check{\tau} P_z^2\right)
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where $\hat{\tau} = \tau m\omega/\hbar$, $\check{\tau} = \tau/(m\omega\hbar)$

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- Representation in flat noncommutative space:

$$X = (1 + \check{\tau} p_{x_0}^2) x_0 + \frac{\theta_1}{\hbar} \left(\check{\tau} p_{x_0}^2 - \hat{\tau} y_0^2\right) p_{y_0}, \qquad P_x = p_{x_0},$$

$$Z = (1 + \check{\tau} p_{z_0}^2) z_0 + \frac{\theta_3}{\hbar} \left(\hat{\tau} y_0^2 - \check{\tau} p_{z_0}^2\right) p_{y_0}, \qquad P_z = p_{z_0},$$

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$$P_y = (1+\hat{\tau}y_0^2)p_{y_0}, \qquad Y = y_0.$$

- Bopp-shift to standard canonical variables:

$x_0 \to x_s - \frac{\theta_1}{\hbar}p_{y_s}$, $y_0 \to y_s$, $z_0 \to z_s + \frac{\theta_3}{\hbar}p_{y_s}$,
$p_{x_0} \to p_{x_s}$, $p_{y_0} \to p_{y_s}$, $p_{z_0} \to p_{z_s}$

- Dyson map: $\eta = \eta_{y_0}\eta_{p_{x_0}}\eta_{p_{z_0}}$

$$\eta_{y_0} = (1+\hat{\tau}y_0^2)^{-1/2}, \quad \eta_{p_{x_0}} = (1+\check{\tau}p_{x_0}^2)^{-1/2}, \quad \eta_{p_{z_0}} = (1+\check{\tau}p_{z_0}^2)^{-1/2}$$

- Hermitian variables:

$$\begin{aligned}
x &:= \eta X \eta^{-1} = \eta_{p_{x_0}}^{-1}\left(x_0 + \frac{\theta_1}{\hbar}\right)\eta_{p_{x_0}}^{-1} - \frac{\theta_1}{\hbar}\eta_{y_0}^{-1}p_{y_0}\eta_{y_0}^{-1} = x^\dagger \\
y &:= \eta Y \eta^{-1} = y_0 = y^\dagger \\
z &:= \eta Z \eta^{-1} = \eta_{p_{z_0}}^{-1}\left(z_0 - \frac{\theta_3}{\hbar}\right)\eta_{p_{z_0}}^{-1} + \frac{\theta_3}{\hbar}\eta_{y_0}^{-1}p_{y_0}\eta_{y_0}^{-1} = z^\dagger \\
p_x &:= \eta P_x \eta^{-1} = p_{x_0} = p_x^\dagger \\
p_y &:= \eta P_y \eta^{-1} = \eta_{y_0}^{-1}p_{y_0}\eta_{y_0}^{-1} = p_y^\dagger \\
p_z &:= \eta P_z \eta^{-1} = p_{z_0} = p_z^\dagger
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p_z &:= \eta P_z \eta^{-1} = p_{z_0} = p_z^\dagger
\end{aligned}$$

- Isospectral Hermitian counterpart:

$$H(X,Y,Z,P_x,P_y,P_z) \neq H^\dagger(X,Y,Z,P_x,P_y,P_z) \Rightarrow h = \eta H \eta^{-1} = h^\dagger$$

- Metric: $\rho = \eta^2$

A particular $\mathcal{PT}_{\theta_\pm}$-symmetric solution

Now starting from a representation:

$$
\begin{aligned}
X &= x_0 - \hat{\tau}\tfrac{\theta_1}{\hbar} y_0^2 p_{y_0} - \hat{\tau}\tfrac{\theta_2}{\hbar} y_0^2 p_{z_0}, & P_x &= p_{x_0}, \\
Z &= z_0 + \hat{\tau}\tfrac{\theta_3}{\hbar} y_0^2 p_{y_0} + \hat{\tau}\tfrac{\theta_2}{\hbar}\tfrac{\theta_3}{\theta_1} y_0^2 p_{z_0}, & P_z &= p_{z_0}, \\
P_y &= p_{y_0} + \hat{\tau} y_0^2 p_{y_0}, & Y &= y_0,
\end{aligned}
$$

yields the closed algebra

$$
\begin{aligned}
[X, Y] &= i\theta_1 (1 + \hat{\tau} Y^2), & [X, Z] &= i\theta_2 (1 + \hat{\tau} Y^2) \\
[X, P_x] &= i\hbar, & [Y, P_y] &= i\hbar (1 + \hat{\tau} Y^2), \\
[Y, Z] &= i\theta_3 (1 + \hat{\tau} Y^2) & [Z, P_z] &= i\hbar
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$$
\begin{aligned}
[X, Y] &= i\theta_1 \left(1 + \hat{\tau} Y^2\right), & [X, Z] &= i\theta_2 \left(1 + \hat{\tau} Y^2\right) \\
[X, P_x] &= i\hbar, & [Y, P_y] &= i\hbar \left(1 + \hat{\tau} Y^2\right), \\
[Y, Z] &= i\theta_3 \left(1 + \hat{\tau} Y^2\right) & [Z, P_z] &= i\hbar
\end{aligned}
$$

for $\theta_1 = -\theta_3$ this is also $\mathcal{PT}_{xz}$-symmetric

Minimal lengths, areas and volumes

Uncertainty relation:

$$\Delta A \Delta B \geq \frac{1}{2} \left| \langle [A, B] \rangle_\rho \right|$$

- For instance for the special solution with $q \to 1$:

$$\Delta X_{\text{min}} = |\theta_1| \sqrt{\hat{\tau} + \hat{\tau}^2 \langle Y \rangle_\rho^2}, \Delta Y_{\text{min}} = 0, \Delta Z_{\text{min}} = |\theta_3| \sqrt{\hat{\tau} + \hat{\tau}^2 \langle Y \rangle_\rho^2},$$

$$\Delta X_{\text{min}} = \hbar \sqrt{\hat{\tau} + \hat{\tau}^2 \langle Y \rangle_\rho^2}, \Delta Y_{\text{min}} = 0, \Delta Z_{\text{min}} = \hbar \sqrt{\hat{\tau} + \hat{\tau}^2 \langle Y \rangle_\rho^2},$$

$$\Delta (P_x)_{\text{min}} = 0, \Delta (P_y)_{\text{min}} = \hbar \sqrt{\hat{\tau} + \hat{\tau}^2 \langle Y \rangle_\rho^2}, \Delta (P_z)_{\text{min}} = 0.$$

Before taking the limit, we obtain for instance:

$$\Delta Y_{\min} = |\hat{\kappa}_6| \sqrt{\frac{1}{2}(q^2 - q^{-2}) + (q - q^{-1})^2 \left(\frac{1}{4\hat{\kappa}_6^2} \langle Y \rangle_\rho^2 + \frac{\hat{\kappa}_6^2}{\hbar^2} \langle P_y^2 \rangle_\rho \right)}$$

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absolute minimal lengths: $(Q := \sqrt{q^2 - q^{-2}})$

$$\Delta X_0 = \frac{1}{2\sqrt{2}} \left| \frac{\theta_1}{\hat{\kappa}_6} \right| Q, \quad \Delta Y_0 = \frac{|\hat{\kappa}_6|}{\sqrt{2}} Q, \quad \Delta Z_0 = \frac{1}{2\sqrt{2}} \left| \frac{\theta_3}{\hat{\kappa}_6} \right| Q$$

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$\Rightarrow$ absolute minimal uncertainty volume:

$$\Delta V_0 = \frac{1}{\sqrt{2}} \left| \frac{\theta_1 \theta_3}{\hat{\kappa}_6} \right| \left(q^2 - q^{-2} \right)^{3/2}$$

- similarly for the momenta

Different types of representations

$$[X, P] = i\hbar\left(1 + \check{\tau} P^2\right)$$

non-Hermitian: $X_{(1)} = (1 + \check{\tau} p^2)x, \quad P_{(1)} = p$

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How are these representations related?

Solvable non-Hermitian potentials

$$H(p)\psi(p) = E\psi(p) \Leftrightarrow -f(p)\psi''(p)+g(p)\psi'(p)+h(p)\psi(p) = E\psi(p)$$

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transformation:

$$\psi(p) = e^{\chi(p)}\phi(p), \quad \chi(p) = \int \frac{f'(p) + 2g(p)}{4f(p)} dp, \quad q = \int \sqrt{f(p)} dp$$

$$\phi(q) = v(q) F[w(q)]$$

general formula for the metric:

$$\rho(p) = \varrho(p) e^{-2 \,\mathrm{Re}\, \chi(p)} \left| v(p) \right|^{-2} \frac{dw}{dp}$$

Swanson model in different representation

On a noncommutative space the model is 2 × non-Hermitian:

$$\begin{aligned} H_{(i)} &= \hbar\omega\left(A^{\dagger}_{(i)}A_{(i)} + \frac{1}{2}\right) + \alpha A_{(i)}A_{(i)} + \beta A^{\dagger}_{(i)}A^{\dagger}_{(i)} \quad \text{for } i = 1,2,3,4 \\ &= \frac{\hbar\omega(1-\tau) - \alpha - \beta}{2m\hbar\omega} P^2_{(i)} + \frac{\Omega m\omega}{2\hbar} X^2_{(i)} + i\left(\frac{\alpha-\beta}{2\hbar}\right)\{X_{(i)}P_{(i)}\} \end{aligned}$$

$A_{(j)} = \left(m\omega X_{(j)} + iP_{(j)}\right)/\sqrt{2m\hbar\omega}$,
$A^{\dagger}_{(j)} = \left(m\omega X_{(j)} - iP_{(j)}\right)/\sqrt{2m\hbar\omega}$,
$\Omega := \alpha + \beta + \hbar\omega$,
$\alpha, \beta \in \mathbb{R}$

Energy:

$$E_n = \frac{1}{4}[(\tau + 2n\tau + 2n^2\tau)\Omega + (2n+1) + \sqrt{4\left(\hbar^2\omega^2 - 4\alpha\beta\right) + \tau\Omega(\tau\Omega - 4\hbar\omega)}]$$

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- representation 1:

$$\psi_n(p) = \frac{1}{\sqrt{N_n}}(1 + \check{\tau}p^2)^{\frac{\beta-\alpha}{2\tau\Omega}-\frac{1}{4}} P^{\mu_-}_{n-\mu_-}\left(\frac{\sqrt{\check{\tau}}p}{\sqrt{1+\check{\tau}p^2}}\right)$$

$$\rho(p) = \sqrt{\check{\tau}}\left(1 + \check{\tau}p^2\right)^{\frac{\alpha-\beta-\tau\Omega}{\tau\Omega}}$$

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- representation 2:

$$\psi_n(p) = \frac{1}{\sqrt{N_n}}\left[\cos\left(\sqrt{\check{\tau}}p\right)\right]^{\frac{\alpha-\beta}{\tau\Omega}+\frac{1}{2}} P^{\mu_-}_{n-\mu_-}\left[\sin\left(\sqrt{\check{\tau}}p\right)\right]$$

$$\rho(p) = \sqrt{\check{\tau}}\left[\cos\left(\sqrt{\check{\tau}}p\right)\right]^{\frac{2(\beta-\alpha)}{\tau\Omega}}$$

restricted momentum $-\pi/2\sqrt{\check{\tau}} \leq p \leq \pi/2\sqrt{\check{\tau}}$

For all four representations:

$$\left\langle \psi_{(i)} \right| F\left(P_{(i)}, X_{(i)}\right) \psi_{(i)} \Big\rangle_{\rho_{(i)}}$$

$$= \frac{1}{N} \int_{-1}^{1} F\left[\frac{z}{\sqrt{\check{\tau}(1-z^2)}}, i\hbar\sqrt{\check{\tau}(1-z^2)}\partial_z\right] \left| P_{m-\mu_-}^{\mu_-}(z)\right|^2 dz$$

Klauder coherent states

$$|J,\gamma,\phi\rangle = \frac{1}{\mathcal{N}(J)}\sum_{n=0}^{\infty}\frac{J^{n/2}\exp(-i\gamma e_n)}{\sqrt{\rho_n}}\,|\phi_n\rangle\,, \qquad J\in\mathbb{R}_0^+,\gamma\in\mathbb{R}$$

with

$$h\,|\phi_n\rangle = \hbar\omega e_n\,|\phi_n\rangle\,, \quad \rho_n := \prod\nolimits_{k=1}^{n} e_k, \quad \mathcal{N}^2(J) := \sum\nolimits_{k=0}^{\infty}\frac{J^k}{\rho_k}$$

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Basis properties:

- continuous in J, γ
- provide a resolution of the identity
- temporary stable
- satisfy action identity

$$\langle J,\gamma,\Phi| H |J,\gamma,\Phi\rangle_\eta = \langle J,\gamma,\phi| h |J,\gamma,\phi\rangle = \hbar\omega J$$

[J.R. Klauder; Annals Phys. 237 (1995) 147]

Generalized Heisenberg's uncertainty relation

For a measurement of two observables A and B we have:

$$\Delta A \Delta B \geq \frac{1}{2} \left| \langle J, \gamma, \Phi | \left[A, B \right] | J, \gamma, \Phi \rangle_{\eta} \right|$$

Uncertainties:

$$\Delta A = \langle J, \gamma, \Phi | A^2 | J, \gamma, \Phi \rangle_{\eta} - \langle J, \gamma, \Phi | A | J, \gamma, \Phi \rangle_{\eta}^2$$

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Ehrenfest theorem

$$i\hbar \frac{d}{dt} \langle J, \gamma + t\omega, \Phi | A | J, \gamma + t\omega, \Phi \rangle_\eta = \langle J, \gamma + t\omega, \Phi | [A, H] | J, \gamma +$$

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Time evolution

$$\exp(-iHt/\hbar) \, | J, \gamma, \Phi \rangle = | J, \gamma + t\omega, \Phi \rangle$$

1D noncommutative harmonic oscillator

$$H = \frac{P^2}{2m} + \frac{m\omega^2}{2} X^2 - \hbar\omega \left(\frac{1}{2} + \frac{\tau}{4} \right)$$

defined on the noncommutative space

$$[X, P] = i\hbar \left(1 + \check{\tau} P^2 \right), \qquad X = (1 + \check{\tau} p^2) x, \qquad P = p$$

1D noncommutative harmonic oscillator

$$H = \frac{P^2}{2m} + \frac{m\omega^2}{2}X^2 - \hbar\omega\left(\frac{1}{2} + \frac{\tau}{4}\right)$$

defined on the noncommutative space

$$[X,P] = i\hbar\left(1 + \check{\tau}P^2\right), \qquad X = (1+\check{\tau}p^2)x, \qquad P = p$$

first order perturbation theory

$$E_n = \hbar\omega e_n = \hbar\omega n\left[1 + \frac{\tau}{2}(1+n)\right] + \mathcal{O}(\tau^2)$$

$$|\phi_n\rangle = |n\rangle - \frac{\tau}{16}\sqrt{(n-3)_4}\,|n-4\rangle + \frac{\tau}{16}\sqrt{(n+1)_4}\,|n+4\rangle + \mathcal{O}(\tau^2)$$

Pochhammer function $(x)_n := \Gamma(x+n)/\Gamma(x)$

$$\rho_n = \frac{1}{2^n}\tau^n n!\left(2 + \frac{2}{\tau}\right)_n \qquad \mathcal{N}^2(J) = e^J\left(1 - \tau J - \frac{\tau}{4}J^2\right) + \mathcal{O}(\tau^2)$$

the K-states saturate the generalized uncertainty relation

$$\Delta X^2 = \langle J,\gamma,\Phi| X^2 |J,\gamma,\Phi\rangle_\eta - \langle J,\gamma,\Phi| X |J,\gamma,\Phi\rangle_\eta^2$$

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Strong disagreement with
[S. Gosh, P. Roy; Phys. Lett. B711(2012) 423]

Ehrenfest theorem:

$$i\hbar \frac{d}{dt} \langle J, \gamma + t\omega, \Phi| \, X \, |J, \gamma + t\omega, \Phi\rangle_\eta = \langle J, \gamma + t\omega, \Phi| \, [X, H] \, |J, \gamma + t\omega, \Phi\rangle_\eta$$

$$= \langle J, \gamma + t\omega, \Phi| \, \frac{i\hbar}{m}(P + \check{\tau} P^3) \, |J, \gamma + t\omega, \Phi\rangle_\eta$$

$$= -i\hbar^{3/2}\sqrt{\frac{2J\omega}{m}} \left[\sin\hat{\gamma} + \tau\left[(J+1)\hat{\gamma}\cos\hat{\gamma} + \frac{1}{2}\sin\hat{\gamma}(2 + J - 3J\cos 2\hat{\gamma})\right]\right]$$

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$$=\ \left\langle J,\gamma+t\omega,\Phi\right|-im\hbar\omega^2\left(X+\frac{\check{\tau}}{2}XP^2+\frac{\check{\tau}}{2}P^2X\right)\left|J,\gamma+t\omega,\Phi\right\rangle_\eta$$

$$=\ -i\sqrt{2Jm(\omega\hbar)^3}\left[\cos\hat{\gamma}+\frac{\tau}{4}\left[(3J+2)\cos\hat{\gamma}-4(J+1)\hat{\gamma}\sin\hat{\gamma}-3J\cos3\hat{\gamma}\right]\right]$$

Fractional revival structure

Given a wave-packet $\psi = \sum c_n \phi_n$ localized at $n = \bar{n}$ with $E_{\bar{n}}$

- revival after classical period $T_{cl} = 2\pi\hbar / |E'_{\bar{n}}|$
- partial revival after p/qT_{rev} with revival time $T_{rev} = 4\pi\hbar / |E''_{\bar{n}}|$

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K-coherent states for HO on noncommutative spacetime

$$|J, \omega t, \phi\rangle = \sum_{n=0}^{\infty} c_n(J) \exp(-itE_n/\hbar) |\phi_n\rangle$$

Weighting function: $c_n(J) = J^{n/2} / \mathcal{N}(J) \sqrt{\rho_n}$

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Mandel parameter:

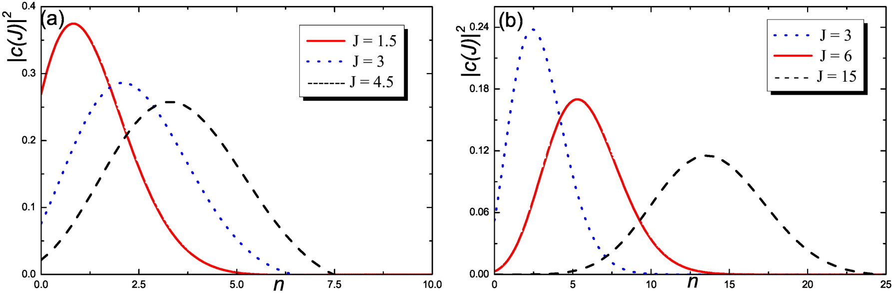
$$Q := \frac{\Delta n^2}{\langle n \rangle} - 1 = -\frac{J\tau}{2} + \mathcal{O}(\tau^2) < 0$$

$$\langle n \rangle = J - \tau \left(J + \frac{J^2}{2} \right) + \mathcal{O}(\tau^2), \ \langle n^2 \rangle = J + J^2 - \tau \left(J + 3J^2 + J^3 \right) + \mathcal{O}(\tau$$

$$\Delta n^2 = \langle n^2 \rangle - \langle n \rangle^2 = J - \tau \left(J + J^2 \right) + \mathcal{O}(\tau^2).$$

$\Rightarrow$ sub-Poissonian statistics

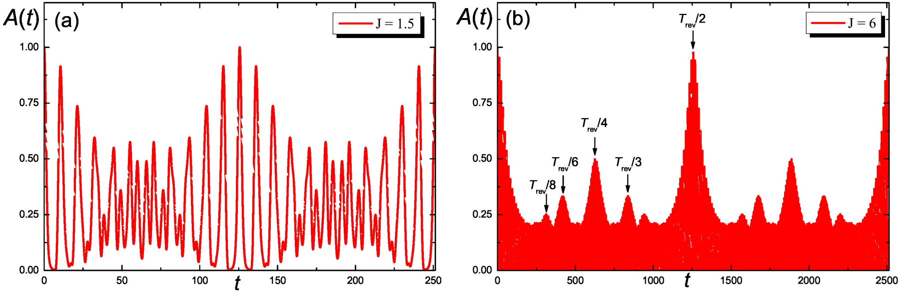
Weighting function



(a) $\tau = 0.1$ with $\langle n \rangle = 1.24, 2.25, 3.04$
(b) $\tau = 0.01$ with $\langle n \rangle = 2.93, 5.76, 13.72$

Autocorrelation function

$$A(t) := |\langle J, \gamma, \phi \, | J, \gamma + t\omega, \phi \rangle|^2 = |\langle J, \gamma, \Phi \, | J, \gamma + t\omega, \Phi \rangle_\eta \, |^2$$



(a) $J = 1.5,\ \tau = 0.1,\ \omega = 0.5,\ \gamma = 0,\ T_{\text{cl}} = 10.05\ \ T_{\text{rev}} = 251.32$
(b) $J = 6,\ \tau = 0.01,\ \omega = 0.5, \gamma = 0,\ T_{\text{cl}} = 11.74\ \ T_{\text{rev}} = 2513.27$

$$T_{\text{cl}} = \frac{2\pi}{\omega} - \frac{\tau}{\omega}(1 + 2J)\pi, \qquad T_{\text{rev}} = \frac{4\pi}{\omega\tau}$$

Q-dependent coherent states

Now consider deformed canonical commutation relations:

$$[X, P] = i\hbar + i\frac{q^2 - 1}{q^2 + 1}\left(m\omega X^2 + \frac{1}{m\omega}P^2\right)$$

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Take

$$X = \alpha\left(A^\dagger + A\right), \qquad \text{and} \qquad P = i\beta\left(A^\dagger - A\right)$$

with $\alpha = 1/2\sqrt{1+q^2}\sqrt{\hbar/(m\omega)}$, $\beta = 1/2\sqrt{1+q^2}\sqrt{\hbar m\omega}$

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Hermitian representation:

$$A = \frac{i}{\sqrt{1-q^2}}\left(e^{-i\check{x}} - e^{-i\check{x}/2}e^{2\tau\check{p}}\right), \; A^\dagger = \frac{-i}{\sqrt{1-q^2}}\left(e^{i\check{x}} - e^{2\tau\check{p}}e^{i\check{x}/2}\right)$$

with $\check{x} = x\sqrt{m\omega/\hbar}$ and $\check{p} = p/\sqrt{m\omega\hbar}$

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Non-Hermitian representation:

$$A = \frac{1}{1-q^2}D_q, \qquad \text{and} \qquad A^\dagger = (1-x) - x(1-q^2)D_q$$

Jackson derivatives $D_q f(x) := [f(x) - f(q^2x)]/[x(1-q^2)]$

For Hermitian representation:

- The uncertainty relations:

$$\Delta X \Delta P|_{|J,\gamma\rangle_q} \geq \frac{1}{2} \left| \left({}_q\langle J,\gamma| \, [X,P] \, |J,\gamma\rangle_q \right)_\eta \right|$$

  are shown to hold, but are saturated only for $t = 0$.

- Infinitely many revival times emerge,
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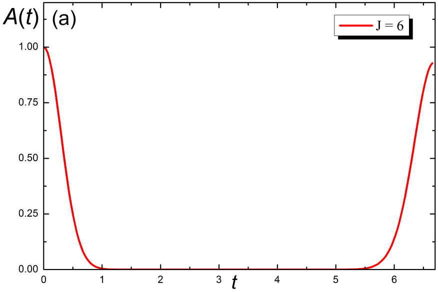
Classical period: $T_{\text{cl}} = 6.65$, $q = e^{-0.005}$, $J = 6$, $\bar{n} = 6.1875$



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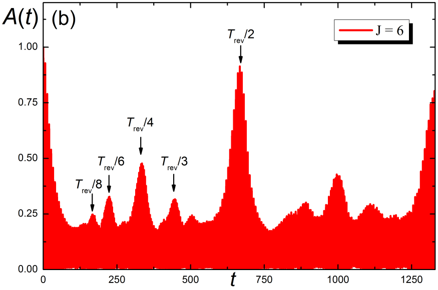
Revival: $T_{\text{rev}} = 1330.19$, $q = e^{-0.005}$, $J = 6$, $\bar{n} = 6.1875$



For Hermitian representation:

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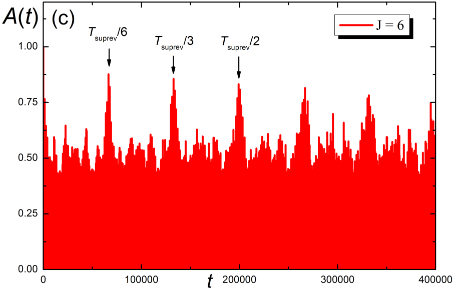
Superrevival: $T_{\text{suprev}} = 3999056$, $q = e^{-0.005}$, $J = 6$, $\bar{n} = 6.1875$



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Thank you for your attention