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CITY UNIVERSITY

London

BSc Degrees in Mathematical Science
Mathematical Science with Statistics
Mathematical Science with Computer Science
Mathematical Science with Finance and Economics
MMath Degrees in Mathematical Science

PART III EXAMINATION

Mathematical Methods

27 May 2003 9:00 am – 11:00 am

Time allowed: 2 hours

Full marks may be obtained for correct answers to THREE questions. Candidates for an Honours Degree may only choose from the first FIVE questions. Candidates for for an Ordinary Degree may select from all SIX questions.

If more than THREE questions are answered, the best THREE marks will be credited.

1. (a) State the convolution theorem for the Laplace transform. Use the convolution theorem to obtain the inverse Laplace transform

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s-1)^4}\right\}.$$

(b) Using the Laplace transform method, find the function $y(t)$ satisfying

$$\frac{d^2y}{dt^2} + 11\frac{dy}{dt} + 28y = g(t)$$

with $y(0) = y'(0) = 0$, where

$$g(t) = \begin{cases} 1; & 0 \le t < 2 \\ 0; & t \ge 2 \end{cases}$$

2. (a) If the sine Fourier transform of $f(x)$ is $F_s(\omega)$, show, stating the conditions that must be satisfied by f and f', that the cosine Fourier transform $F_c(f') = \omega F_s(\omega) - f(0)$. Show also that $F_s(f') = -\omega F_c(\omega)$. By taking the cosine Fourier transform of the identity

$$\frac{d^2(e^{-ax})}{dx^2} = a^2 e^{-ax},$$

find the cosine Fourier transform of the function $f(x) = e^{-ax}$, where a is a positive number.

(b) Determine the cosine Fourier transform of the function $f(x) = xe^{-x}$. Hence evaluate the integral

$$\int_0^\infty \frac{(1-\omega^2)\cos(\omega x)}{(1+\omega^2)^2}\, d\omega,$$

where $x \ge 0$.

Turn over ...

3. A function of two variables $u(x,t)$ satisfies the partial differential equation

$$\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$$

in the region $x \geq 0$, $t \geq 0$.

Using Laplace transforms with respect to the variable t, find $u(x,t)$ satisfying the condition

$$u(x,0) = 6e^{-3x}$$

given that $u(x,t)$ is bounded for $x \geq 0$, $t \geq 0$.

4. (a) The general equation of a circle or straight line in the (x,y)-plane has the form $A(x^2+y^2)+Bx+Cy+D=0$, where A, B, C and D are real numbers. Writing $z = x+iy$, express the equation of the circle (line) in terms of z and $z^* = x - iy$. Hence prove that the mapping $w = z^{-1}$ maps every circle or straight line in the complex z-plane onto a circle or a line in the w-plane.

   (b) Find the linear fractional transformation that maps the points $z_1 = 0$, $z_2 = 1$, $z_3 = \infty$ in the complex z-plane onto $w_1 = -1$, $w_2 = -i$, $w_3 = 1$ in the complex w-plane.

   What is the region in the z-plane that is mapped by such a linear fractional transformation onto $|w| = 2$?

5. (a) Show that the mapping $w = (1+z)/(1-z)$ maps the unit disc $|z| \leq 1$ in the complex z-plane onto the right-hand half $Re\ w \geq 0$ of the complex w-plane.

(b) Show that the function $\phi(x, y) = a + b\ Arg(z)$, where $z = x + iy$ and a, b are arbitrary constants, satisfies Laplace's equation $\nabla^2\phi = 0$.

Two metallic plates perpendicular to the (x, y)-plane intersect the (x, y)-plane along the lines $\arg(z) = \pi/2$ and $\arg(z) = -\pi/2$ as shown in Fig.1(a). Given that the lower plate is kept at a constant

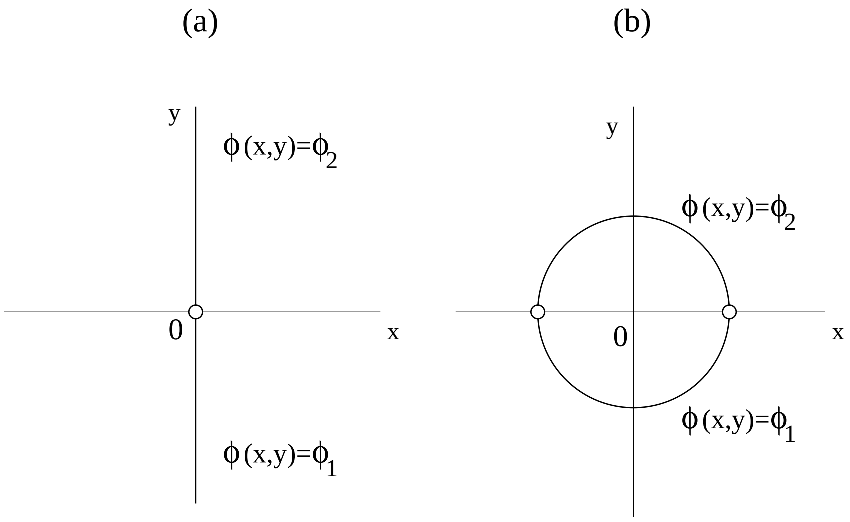

Figure 1:

potential ϕ_1 and the upper plate at a potential ϕ_2, find the potential $\phi(x, y)$ between the two plates.

(c) Two semi-circular metallic plates perpendicular to the (x, y)-plane intersect the (x, y)-plane in a circle $|z| = 1$, as shown in Fig. 1(b). The lower plate is kept at a constant potential ϕ_1 and the upper plate at a potential ϕ_2. Using the mapping defined in (a), find the potential $\phi(x, y)$ between the two plates.

The following question may be attempted only by candidates for the Ordinary Degree

6. (a) Find the constant α so that

$$v(x, y) = \frac{y}{x^2 + \alpha y^2}$$

is a harmonic function. Hence find an analytic function $f(z) = u(x, y) + iv(x, y)$.

(b) By using the definition of the Laplace transform, find the Laplace transform of the following:

(i) $f(t) = t^2$;

(ii) $f(t) = \sin at$, where a is a constant.

Internal Examiner:	Professor J. Mathon
External Examiners:	Professor D.J. Needham
	Professor M.E. O'Neill