1. (a) State the convolution theorem for the Laplace transform. Use the convolution theorem to obtain the inverse Laplace transform

$$\mathcal{L}^{-1}\{\frac{s}{(s^2+1)^2}\}.$$

(b) Find the Laplace transform $Y(s)$ of the function $y(t)$ which satisfies the initial-value problem

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 4\frac{\mathrm{d}y}{\mathrm{d}t} + 3y = u(t) - u(t-1), \qquad y(0) = y'(0) = 0.$$

Here, the unit step function $u(x) = 0$ if $x < 0$, and $= 1$ if $x > 0$. Find the solution $y(t)$ by using the information in the table of Laplace transforms below.

$f(t)$	$F(s) = \int_0^\infty f(t)e^{-st}\,\mathrm{d}t$
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$
$f(t-a)u(t-a)$	$e^{-as}F(s)$
$e^{at}f(t)$	$F(s-a)$
1	$1/s$
t^n	$n!/s^{n+1}$
e^{-at}	$1/(s+a)$
$\sin at$	$a/(s^2+a^2)$
$\cos at$	$s/(s^2+a^2)$

Turn over ...

2. (a) Using the definition of the Fourier transform

$$\mathcal{F}(f) = \hat{f}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(x)e^{-i\omega x}\,\mathrm{d}x$$

find the Fourier transform of

$$f(x) = e^{-a|x|}, \quad \text{where} \quad a > 0.$$

Hence, or otherwise, find the Fourier transform of

$$f(x) = \frac{1}{x^2 + b^2},$$

where b is a real number.

(b) Assuming that $f(x)$ is continuous and absolutely integrable on the x axis, $f'(x)$ is piecewise continuous on the x axis and $lim_{x\to\pm\infty} f(x) = 0$, show that

$$\mathcal{F}\left(\frac{df}{dx}\right) = i\omega\mathcal{F}(f).$$

(c) A potential, ϕ, satisfies Laplace's equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi = 0$$

in the region $-\infty < x < \infty$, $0 \leq y < \infty$. By taking a Fourier transform with respect to x reduce this to an ordinary differential equation for the transform of the potential, $\hat{\phi}$.

If ϕ satisfies the boundary conditions

$$\phi(x,0) = \begin{cases} 1, & |x| < 1, \\ 0, & |x| \geq 1 \end{cases}$$

$$\phi(x,y) \to 0 \quad \text{as} \quad x \to \pm\infty, \qquad \phi(x,y) \to 0 \quad \text{as} \quad y \to \infty,$$

find the conditions that $\hat{\phi}$ satisfies at $y = 0$ and as $y \to \infty$ and hence find $\hat{\phi}$. *Do not try to invert this Fourier transform to find the potential.*

Turn over ...

3. (a) If the sine Fourier transform of $f(x)$ is $F_s(\omega)$, show, stating the conditions that must be satisfied by f and f', that the cosine Fourier transform $F_c(f') = \omega F_s(\omega) - f(0)$. Show also that $F_s(f') = -\omega F_c(\omega)$.

(b) By taking the cosine Fourier transform of the identity

$$\frac{d^2(e^{-ax})}{dx^2} = a^2 e^{-ax},$$

find the cosine Fourier transform of the function $f(x) = e^{-ax}$, where a is a positive number.

(c) Determine the cosine Fourier transform of the function $f(x) = xe^{-x}$. Hence evaluate the integral

$$\int_0^\infty \frac{(1-\omega^2)\cos(\omega x)}{(1+\omega^2)^2}\, d\omega,$$

where $x \geq 0$.

4. (a) Find a mapping that maps the angular region D in the complex plane $z = x+iy$ defined by $-\pi/6 \leq arg(z) \leq \pi/6$ onto the half-plane $u \geq 0$ of the complex plane $w = u + iv$.

(b) Find also the linear fractional transformation that maps the points $z = 0,\ i,\ \infty$ lying on the imaginary axis of the complex z plane onto the points $w = -1,\ i,\ 1$ lying on the unit disc in the complex w plane.

(c) Hence determine a conformal mapping that maps the angular region D onto the unit disc $\mid w \mid \leq 1$.

Turn over ...

5. (a) Show that the mapping $w = (z - b)/(bz - 1)$, where $b < 1$ is a real number, maps the unit circle $|\, z \,| = 1$ in the complex z-plane onto the unit circle $|\, w \,| = 1$ in the w-plane.

(b) Two non-coaxial metallic cylinders perpendicular to the (x, y) plane intersect it in two circles shown in Fig.1. The outer cylinder is kept

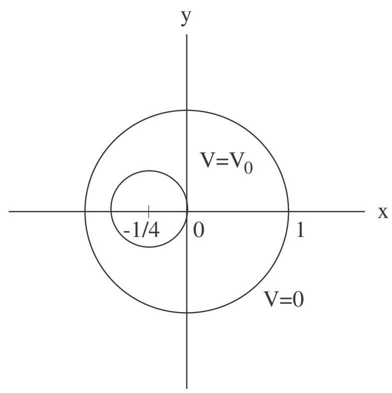

Figure 1:

at zero potential and the inner cylinder at a potential $V_0 = const.$ Determine the parameter b in the mapping $w = (z - b)/(bz - 1)$ so that the point $z = 0$ is mapped onto a real number r such that $0 < r < 1$ and the smaller of the two circles in Fig.1 is mapped onto a circle in the w-plane with centre at $w = 0$ and radius r.

(c) Using the result that the potential V of two coaxial cylinders in the w-plane is given by $V = a \ln |\, w \,| + k$, where a and k are real numbers, determine the potential of the two non-coaxial cylinders in the z-plane.

Turn over ...

6. (a) Find the constant a so that

$$v(x,y) = \frac{y}{x^2 + ay^2}$$

is a harmonic function. Hence find an analytic function $f(z) = u(x,y) + iv(x,y)$.

(b) Using the definition of a Fourier transform

$$F(f) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx$$

find the Fourier transform of the function

$$f(x) = \begin{cases} x, & |x| < a, \\ 0, & a < |x|. \end{cases}$$

Internal Examiner: Professor J. Mathon
External Examiner: Professor M.A. O'Neill and Professor D.J. Needham