1. (a) State the convolution theorem for the Laplace transform. Use the convolution theorem to obtain the inverse Laplace transform

$$\mathcal{L}^{-1}\{\frac{1}{s(s-4)^2}\}.$$

(b) Using the Laplace transform method, find the function $y(t)$ satisfying

$$\frac{d^2y}{dt^2} + 11\frac{dy}{dt} + 28y = g(t)$$

with $y(0) = y'(0) = 0$, where

$$g(t) = \begin{cases} 1; & 0 \leq t < 2 \\ 0; & t \geq 2 \end{cases}$$

2. (a) Show that the sine Fourier transform $F_s(\lambda)$ of the function $f(x) = e^{-kx}$ $(k > 0)$ is given by

$$F_s(\lambda) = \frac{\lambda}{k^2 + \lambda^2}$$

.

(b) By differentiating $F_s(\lambda)$ with respect to the parameter k, find the sine Fourier transform of the function xe^{-kx}.

(c) Using the result in (b) evaluate the integral

$$\int_0^\infty \frac{\lambda \sin \lambda \, d\lambda}{(1+\lambda^2)^2}.$$

Turn over ...

3. A function of two variables $u(x,t)$ satisfies the partial differential equation

$$\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$$

in the region $x \geq 0$, $t \geq 0$.

Using a Laplace transform with respect to the variable t, find $u(x,t)$ satisfying the condition

$$u(x,0) = 6e^{-3x}$$

given that $u(x,t)$ is bounded for $x \geq 0$, $t \geq 0$.

4. (a) The general equation of a circle or straight line in the (x,y) plane has the form $A(x^2+y^2)+Bx+Cy+D=0$, where A, B, C and D are real numbers. Writing $z = x+iy$, express the equation of the circle (line) in terms of z and $z^* = x - iy$. Hence prove that the mapping $w = z^{-1}$ maps every circle or straight line in the complex z-plane onto a circle or a line in the w-plane.

   (b) Find the linear fractional transformation that maps the points $z_1 = 0$, $z_2 = 1$, $z_3 = \infty$ in the complex z-plane onto $w_1 = -1$, $w_2 = -i$, $w_3 = 1$ in the complex w-plane.

   What is the region in the z-plane that is mapped by such a linear fractional transformation onto $\mid w \mid = 2$?

Turn over ...

5. (a) Show that the mapping $w = (1+z)/(1-z)$ maps the unit disc $|z| \leq 1$ in the complex z-plane onto the right-hand half $Re\ w \geq 0$ of the complex w-plane.

(b) Show that the function $\phi(x,y) = a + b\ Arg(z)$, where $z = x + iy$ and a, b are arbitrary constants, satisfies Laplace's equation $\nabla^2\phi = 0$.

Two metallic plates perpendicular to the (x,y)-plane intersect the (x,y)-plane along the lines $Arg(z) = \pi/2$ and $Arg(z) = -\pi/2$ as shown in Fig.1a. Given that the lower plate is kept at a constant

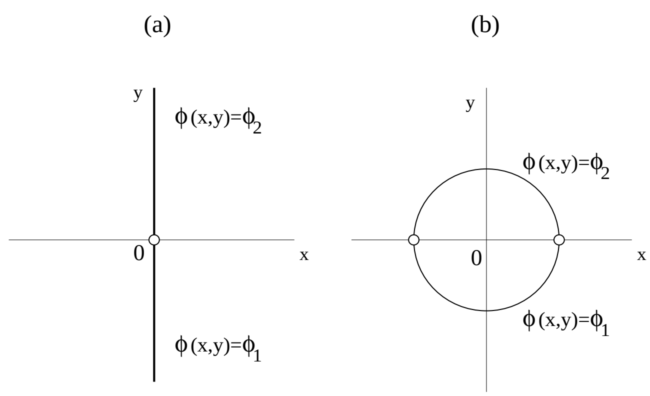

Figure 1:

potential ϕ_1 and the upper plate at a potential ϕ_2, find the potential $\phi(x,y)$ in the region $x \geq 0$.

(c) Two semi-circular metallic plates perpendicular to the (x,y)-plane intersect the (x,y)-plane in a circle $|z| = 1$, as shown in Fig. 1b. The lower plate is kept at a constant potential ϕ_1 and the upper plate at a potential ϕ_2. Using the mapping defined in (a), find the potential $\phi(x,y)$ between the two plates.

Internal Examiner: Professor J. Mathon
External Examiners: Prof. D.J. Needham
Prof. M.E. O'Neill