**MA 3605 Mathematical Methods Autumn 2005**
**Exam**

1. COMPLEX VARIABLES AND BOUNDARY VALUE PROBLEMS. Consider the unit disc

$$D = \{z = x + iy : x^2 + y^2 \leq 1\}$$

in the complex $z$-plane centred at the origin $z = 0$. Find a solution $\phi(x, y)$ to Laplace's equation

$$\Delta\phi(x, y) = (\partial_x^2 + \partial_y^2)\phi(x, y) = 0$$

subject to the Neumann boundary condition

$$\frac{\partial\phi}{\partial\vec{N}}(x, y) = 0 \quad \text{for all } (x, y) \in \partial D$$

by following the different steps given below. Here $\vec{N}$ is the normal vector with respect to the circle

$$\partial D = \{z = x + iy : x^2 + y^2 = 1\} \ .$$

(a) CONFORMAL MAPPING. State the definition of a conformal map. What properties guarantee that a map $f$ is conformal? Consider the complex function

$$w = u + iv = f(z) = i\,\frac{1 - z}{1 + z}$$

and determine its real and imaginary part, i.e. find the functions $u(x, y) = \operatorname{Re} f(x + iy)$ and $v(x, y) = \operatorname{Im} f(x + iy)$, respectively. Use your result to argue that $f$ maps the unit disc $D$ into the upper-half of the complex $w$-plane,

$$R = \{u + iv = w \in \mathbb{C} : v \geq 0 \text{ and } -\infty < u < \infty\} \ .$$

Is the map conformal inside of $D$? Where are its critical and singular points if it has any?

(b) HARMONIC CONJUGATE & BOUNDARY CONDITIONS. Compute the real and imaginary part of $\sinh w$ with $w = u + iv$, where $u = \operatorname{Re} w$, $v = \operatorname{Im} w \in \mathbb{R}$. Argue without computation that $\varphi(u, v) = \operatorname{Re}\sinh w$ and $\psi(u, v) = \operatorname{Im}\sinh w$ are harmonic functions on $R$. Show that $\varphi$ satisfies the von Neumann boundary condition

$$\frac{\partial\varphi}{\partial\vec{n}}(u, 0) = 0,$$

where $\vec{n}$ is the normal vector with respect to the real line $\partial R = \{w = u + iv : -\infty < u < \infty,\ v = 0\}$ in the $w$-plane.

(c) COMPOSITE SOLUTION. Using the results from (a) and (b) state the resulting solution $\phi$ to Laplace's equation and the above von Neumann boundary condition on $\partial D$ in terms of the normal vector $\vec{N}$ of $\partial D$. Sketch the normal vector $\vec{N}$ and find a parametrization for it along the circle $\partial D$. Hence, express the directional derivative $\partial\phi/\partial\vec{N}$ in terms of the gradient $\nabla\phi$.

2. FOURIER TRANSFORMATION & THE WAVE EQUATION.

(a) State the definition of the Fourier transform and its existence conditions. Let $f(x)$ be a continuously differentiable function with Fourier transform $\hat{f}(k)$. Compute the Fourier transform of its derivative $f'(x)$ in terms of $\hat{f}(k)$.

(b) Consider the partial differential equation

$$\partial_t^2 u(x,t) = c^2 \partial_x^2 u(x,t), \qquad -\infty < x < \infty, \quad t > 0$$

with initial conditions

$$u(x,0) = 0 \qquad \text{and} \qquad \partial_t u(x,0) = f'(x) \ .$$

Apply Fourier transformation to obtain an ordinary differential equation for the Fourier transform $\hat{u}(k,t)$ of $u(x,t)$. Solve this ordinary differential equation via Laplace transfomation to find the final solution $u(x,t)$ in terms of the function $f(x)$ using the result from (a).

3. FOURIER SERIES & THE HEAT EQUATION.

(a) Let $f(x)$ be even and continuous on the interval $-L < x < L$ and assume that it is $2L$-periodic, $f(x) = f(x+2L)$. Write down the non-vanishing terms in the Fourier series expansion of $f(x)$. State the explicit form of the Fourier coefficients. Now do the same when $f(x)$ is odd.

(b) Solve the partial differential equation

$$\partial_t u(x,t) = \partial_x^2 u(x,t), \qquad 0 < x < L, \quad t > 0$$

with the boundary conditions

$$u(0,t) = u(L,t) = 0, \qquad t \geq 0,$$

and the initial condition

$$u(x,0) = x \ .$$

Extend $u(x,t)$ to the interval $-L < x < L$ by assuming either that it is even or odd. Which of these two choices is in accordance with the boundary conditions? Then expand $u(x,t)$ into a Fourier series and derive an ordinary differential equation for the Fourier coefficients $u_n(t)$ of $u(x,t)$. Determine the latter via Laplace transformation. Express the final solution $u(x,t)$ in terms of the Fourier coefficients of the function $f(x) = x$.

4. LAPLACE TRANSFORMS.

(a) Give the definition and existence conditions of the Laplace transform $\mathcal{L}$ of a function $f$. State the convolution theorem for the Laplace transform. Use the convolution theorem to compute

$$\mathcal{L}^{-1}[\frac{1}{s^2(s^2+\omega^2)}] \ .$$

(b) Prove as a special case of Heaviside's shift theorem,

$$\mathcal{L}[H(t-a)f(t-a)](s) = e^{-as}F(s), \quad a > 0, \quad F = \mathcal{L}f, \quad H(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$

that the Laplace transform of the Heaviside step function $H$ is

$$\mathcal{L}[H(t-a)](s) = \frac{e^{-as}}{s} \, .$$

(c) Use the previous results to solve the initial value problem

$$f''(t) + \omega^2 f(t) = H(t-a), \quad f(0) = f'(0) = 0, \quad a > 0 \, .$$

5. Laplace transforms and systems of linear differential equations. Consider the differential equation

$$f''' - 27f = g, \qquad f(0) = f'(0) = f''(0) = 0$$

with $g$ being constant.

(a) Solve the above differential equation via Laplace transformation and making a partial fraction expansion.

(b) Reformulate the differential equation as a linear system of differential equations $x' = Ax + y$ in normal form (do not solve it). State in abstract terms the form of the general solution. By comparing results with part (a) of the problem (or otherwise) state the eigenvalues of the matrix $A$.

You may assume the following Laplace transforms to be known.

| $f(t)$ | $F(s) = \mathcal{L}f(s)$ |
|---|---|
| 1 | $1/s$ |
| $e^{at}$ | $\frac{1}{s-a}$, $s > a$ |
| $\sin at$ | $\frac{a}{s^2+a^2}$ |
| $\cos at$ | $\frac{s}{s^2+a^2}$ |
| $\sinh at$ | $\frac{a}{s^2-a^2}$, $s > a$ |
| $\cosh at$ | $\frac{s}{s^2-a^2}$, $s > a$ |

**SOLUTION**

1. SOLVING A NEUMANN PROBLEM ON THE UNIT DISC. (**total: 33 pts**)

(a) A conformal map preserves angles, magnitude and direction. A map $f$ is conformal in an open set $U$ if it is analytic in $U$ and it's derivative is nonzero. (**3 points**) The real and imaginary part of the given function $f$ are

$$\begin{aligned} u(x,y) &= \operatorname{Re} f(x+iy) = \frac{2y}{(x+1)^2+y^2}, && \textbf{(2 points)} \\ v(x,y) &= \operatorname{Im} f(x+iy) = \frac{1-x^2-y^2}{(x+1)^2+y^2} && \textbf{(2 points)} . \end{aligned}$$

Since $z = x+iy$ lies on the unit disc, we must have $|z|^2 = x^2+y^2 \leq 1$. From this we infer that $v \geq 0$ (**3 points**).
A point where $f$ is not analytic is called singular and a point where its derivative is zero is called critical. To determine these points we compute the derivative

$$f'(z) = -\frac{2i}{(1+z)^2}$$

which shows that there are no critical points but a singular point at $z=-1$. Thus, $f$ is conformal in the complex plane with the exception of $z=-1$. (**5 points**)

(b) Decomposition into real and imaginary part yields

$$\sinh(u+iv) = \sinh u \cos v + i \cosh u \sin v = \varphi(u,v) + i\psi(u,v) \qquad \textbf{(5 pts)} .$$

Both maps are harmonic in $R$ since $\sinh w$ is analytic on the entire complex plane (Cauchy-Riemann equations).
The normal vector is the unit vector which is perpendicular to the boundary and by convention points outward. Since the boundary $\partial R$ is the real line, the normal vector in the present case is

$$\vec{n} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \qquad \textbf{(1 pt)}$$

and we therefore have

$$\frac{\partial \varphi}{\partial \vec{n}}(u,0) = \nabla\varphi(u,0)\cdot\vec{n} = \partial_v \varphi(u,0) = 0 \qquad \textbf{(1 pt)} .$$

(c) The composite solution $\phi = \varphi \circ f$ reads

$$\begin{aligned} \phi(x,y) &= \varphi\left[u(x,y), v(x,y)\right] \\ &= \sinh\left(\frac{2y}{(x+1)^2+y^2}\right)\cos\left(\frac{1-x^2-y^2}{(x+1)^2+y^2}\right) && \textbf{(3 pts)} . \end{aligned}$$

The normal vector $\vec{N}$ is the unit radial vector,

$$\vec{N} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \quad -\pi < \theta < \pi, \qquad \textbf{(3 pts)}$$

whence

$$\frac{\partial \phi}{\partial \vec{N}}(\cos\theta, \sin\theta) = \cos\theta\ \partial_x\phi(\cos\theta,\sin\theta) + \sin\theta\ \partial_y\phi(\cos\theta,\sin\theta) = 0 \quad \textbf{(3 pts)}\ .$$

The point $\theta = \pi$ is excluded as it is singular (see above).

2. FOURIER TRANSFORM. (**total: 33 pts**)

(a) In the lecture the Fourier transform of a function $f$ was defined with the convention

$$[\mathcal{F}f](k) = \hat{f}(k) = \int_{-\infty}^{\infty} f(x)e^{-2\pi ikx}dx \qquad \textbf{(2 pts)}\ .$$

The Fourier transform exists if

- $\int_{-\infty}^{\infty} |f(x)|dx < \infty$ ( **2pts**)
- $f$ has only a finite number of discontinuities (**2 pts**)
- $f$ is of bounded variation (**2 pts**)
  This is slightly more restrictive than the Lipshitz condition which has not been discussed in the lecture.

$$[\mathcal{F}f'](k) = e^{-2\pi ikx}f(x)|_{-\infty}^{\infty} + 2\pi ik\int_{-\infty}^{\infty} f(x)e^{-2\pi ikx}dx = 2\pi ik\ \hat{f}(k) \quad \textbf{(2 pts)}$$

(b) Employing the Fourier transform we find the ODE

$$\partial_t^2\hat{u}(k,t) = -(2\pi kc)^2\hat{u}(k,t) \qquad \textbf{(3 pts)}$$

with initial conditions

$$\hat{u}(k,0) = 0 \qquad \text{and} \qquad \partial_t\hat{u}(k,0) = 2\pi ik\ \hat{f}(k) \qquad \textbf{(2 pts)}\ .$$

Introducing the Laplace transform

$$\hat{U}(k,s) = \int_0^{\infty} e^{-st}\hat{u}(k,t)dt$$

the ODE leads to the identity

$$\hat{U}(k,s) = \frac{s\ \hat{u}(k,0) + \partial_t\hat{u}(k,0)}{s^2 + (2\pi kc)^2} = \frac{2\pi ik\ \hat{f}(k)}{s^2+(2\pi kc)^2} \qquad \textbf{(5 pts)}\ .$$

The inverse Laplace transform gives

$$\hat{u}(k,t) = i\hat{f}(k)\frac{\sin 2\pi kct}{c} \qquad \textbf{(5 pts)}\ .$$

From this the inverse Fourier transform is easily computed to

$$u(x,t) = \int_{-\infty}^{\infty}\hat{u}(k,t)e^{2\pi ikx}dk = \frac{f(x+ct)-f(x-ct)}{2c} \qquad \textbf{(8 pts)}\ .$$

3. FOURIER SERIES. (**total: 33 pts**)

(a)

$$f \text{ even} \quad : \quad f(x) = \frac{f_0}{2} + \sum_{n=1}^{\infty} f_n \ \cos(\frac{\pi n}{L}x), \quad f_n = \frac{2}{L}\int_0^L \cos(\frac{\pi n}{L}x) f(x) dx \ (\mathbf{3\ pts})$$

$$f \text{ odd} \quad : \quad f(x) = \sum_{n=1}^{\infty} f_n \ \sin(\frac{\pi n}{L}x), \quad f_n = \frac{2}{L}\int_0^L \sin(\frac{\pi n}{L}x) f(x) dx \ (\mathbf{3\ pts})$$

(b) The boundary conditions can be satisfied by extending $u(x,t)$ to an odd function, $u(x,t) := -u(-x,t)$ for $-L < x < 0$. Thus, the formal Fourier series for $u(x,t)$ reads

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) \ \sin(\frac{\pi n}{L}x) \qquad (\mathbf{1\ pt}) \ .$$

Insertion into the heat equation leads to the ODE

$$\partial_t u_n(t) = -(\pi n/L)^2 u_n(t), \qquad u_n(0) = f_n \qquad (\mathbf{5\ pts}) \ .$$

Solving this equation via Laplace transformation gives

$$U_n(s) = \frac{f_n}{s + (\pi n/L)^2}, \qquad U_n(s) = \int_0^{\infty} e^{-st} u_n(t) dt \qquad (\mathbf{5\ pts}) \ .$$

The inverse transform is calculated to

$$u_n(t) = f_n \ e^{-(\frac{\pi n}{L})^2 t} \qquad (\mathbf{3\ pts})$$

and it remains to determine the Fourier coefficients of $f(x) = x$ on the interval $[-L, L]$. (In accordance with our earlier extension of $u$ we also extend $x$ to be odd.) Thus,

$$\begin{aligned} f_n &= \frac{2}{L}\int_0^L \sin(\frac{\pi n}{L}x) x dx \\ &= -\frac{2x\cos(\frac{\pi n}{L}x)}{\pi n}\Big|_0^L + \frac{2}{\pi n}\int_0^L \cos(\frac{\pi n}{L}x) dx = (-)^{n+1}\frac{2L}{\pi n} \qquad (\mathbf{10\ pts}) \end{aligned}$$

and

$$u(x,t) = -2L \sum_{n=1}^{\infty} (-1)^n e^{-(\frac{\pi n}{L})^2 t} \ \frac{\sin(\frac{\pi n}{L}x)}{\pi n} \qquad (\mathbf{3\ pts}) \ .$$

4. LAPLACE TRANSFORMS. (**total: 33 pts**)

(a) The Laplace transform of a function $f(t)$ is defined as

$$F(s) = \mathcal{L}f(s) = \int_0^{\infty} e^{-st} f(t) dt \qquad (\mathbf{1\ pt})$$

and exists for $s > a$ if

- $f$ is piecewise continuous ( **1pt**)
- and (at most) of exponential order $a \geq 0$, i.e. there exists a $t_0 > 0$ such that $|f(t)| \leq$const.$e^{at}$ (**1 pt**)

The convolution theorem for the Laplace transform states that for two functions $f(t), g(t)$ whose Laplace transforms $F(s), G(s)$ exist, one has

$$\mathcal{L}[f * g](s) = F(s)G(s), \qquad f * g(t) = \int_0^t f(t-x)g(x)dx \qquad (\mathbf{2\ pts})\ .$$

Setting

$$F(s) = 1/s^2 \qquad \text{and} \qquad G(s) = \frac{1}{s^2+\omega^2}$$

one deduces that

$$f(t) = t \qquad \text{and} \qquad g(t) = \frac{\sin \omega t}{\omega} \qquad (\mathbf{2\ pts})\ .$$

Hence,

$$\begin{aligned} \mathcal{L}^{-1}[\frac{1}{s^2(s^2+\omega^2)}] &= \int_0^t (t-x)\frac{\sin \omega x}{\omega}dx = -\left.\frac{t\cos \omega x}{\omega^2}\right|_0^t - \int_0^t \frac{x \sin \omega x}{\omega}dx \\ &= \frac{t}{\omega^2}(1-\cos \omega t) + \frac{t\cos\omega t}{\omega^2} - \frac{\sin \omega t}{\omega^3} = \frac{\omega t - \sin \omega t}{\omega^3} \qquad (\mathbf{8\ pts})\ . \end{aligned}$$

(b) Setting $f(t) = 1$ and recalling that $\mathcal{L}[1] = 1/s$ the result follows. (**3 pts**)

(c) Laplace transformation of the ODE yields

$$F(s) = \frac{e^{-as}}{s(s^2+\omega^2)} \qquad (\mathbf{5\ pts})$$

as the transform for the solution $f(t)$. The inverse transform can be easily computed by using the result from $(a)$ and the property

$$[\mathcal{L}g'](s) = sG(s) - g(0)$$

with $G(s) = 1/[s^2(s^2+\omega^2)]$. The result reads

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}[e^{-as}\ s\ \frac{1}{s^2(s^2+\omega^2)}] = H(t-a)\ \left.\frac{d}{dt'}\frac{\omega t' - \sin \omega t'}{\omega^3}\right|_{t'=t-a} \\ &= H(t-a)\frac{1-\cos[\omega(t-a)]}{\omega^2} \qquad (\mathbf{10\ pts})\ . \end{aligned}$$

5. Systems of linear differential equations. (**total: 33 pts**)

(a) Using that $s^3 - 27 = (s-3)(s^2+3s+9)$ (**5 pts**) we obtain after Laplace transformation and a partial fraction expansion

$$\begin{aligned} F(s) &= \frac{g}{s(s-3)(s^2+3s+9)} \qquad (\mathbf{5\ pts}) \\ &= -\frac{g}{27s} + \frac{g}{81(s-3)} + \frac{g(2s+3)}{81(s+3/2)^2+27/4} \qquad (\mathbf{5\ pts}) \end{aligned}$$

The inverse transform is then obtained using the shift theorem

$$\mathcal{L}[e^{at} f(t)] = F(s-a)$$

and the known transform of cos,

$$f(t) = \frac{g}{81}\left(e^{3t} + 2e^{-3t/2}\cos[\frac{3\sqrt{3}t}{2}] - 3\right) \qquad (\mathbf{5\ pts})\ .$$

(b) The differential equation can be rewritten as

$$x'(t) = Ax(t) + y,$$

where

$$x = \begin{pmatrix} f \\ f' \\ f'' \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 27 & 0 & 0 \end{pmatrix}, \qquad y = \begin{pmatrix} 0 \\ 0 \\ g \end{pmatrix} \qquad (\mathbf{3\ pts})\ .$$

The general solution to such a system of linear differential equations takes the form

$$x(t) = \sum_{n=1}^{3} v_n e^{\alpha_n t} - A^{-1}y \qquad (\mathbf{5\ pts})$$

with $(v_n, \alpha_n)$ being the eigenvectors and the corresponding eigenvalues of the matrix $A$. The latter is invertible as $\det A = 27$. Thus, comparing results with part (a) we must have

$$\alpha_1 = 3, \qquad \alpha_2 = -\frac{3}{2}(1 - i\sqrt{3}), \qquad \alpha_3 = -\frac{3}{2}(1 + i\sqrt{3}) \qquad (\mathbf{5\ pts})\ .$$

One extra point is awarded to those students who solve three problems correctly.