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CITY UNIVERSITY

London

BSc Degrees in Mathematical Science
Mathematical Science with Statistics
Mathematical Science with Computer Science
Mathematical Science with Finance and Economics
MMath Degrees in Mathematical Science

PART III EXAMINATION

Mathematical Methods

25th of May 2007

Time allowed: 2 hours

Full marks may be obtained for correct answers to THREE of the FIVE questions.

If more than THREE questions are answered, the best THREE marks will be credited.

Turn over . . .

1) i) What is meant by a conformal map and an analytic function? State a theorem which relates conformal maps and analytic functions.

ii) Specify as a particular conformal transformation a linear fractional transformation $w = f(z)$, which maps the points $z_1 = 1 + i$,

$z_2 = 3, z_3 = -2$ in the z-plane onto $w_1 = 1, w_2 = 0, w_3 = -1$ in the w-plane. Is this map unique?

iii) Find a conformal map $w = f(z)$ which maps the wedge region in the z-plane

$$\mathcal{W} = \{r, \theta : r \in \mathbb{R}^+, 0 \leq \theta < \frac{\pi}{4}\}$$

onto the unit disc $|w| \leq 1$. Draw a figure and indicate the corresponding regions including some characteristic points! Which theorem guarantees that such map exits? Is this map unique?

2) i) Provide a definition for a linear fractional transformation $T(z)$. Show that it can be written as

$$T(z) = \frac{a}{c} + \frac{b - ad/c}{cz + d} \quad \text{for } c \neq 0,$$

such that it may be understood as a composition of more elementary transformations, i.e. a translation f_T^{Δ}, a rotation f_R^{λ} and the inversion map f_I. Express $T(z)$ in terms of these maps. Provide also the decomposition of $T(z)$ into these type of maps for the case $c = 0$.

ii) Provide a definition for the cross ratio T_c of the points (z_1, z_2, z_3, z_4). Use the decomposition of $T(z)$ from i) to demonstrate that the cross ratio is an invariant of the linear fractional transformation.

iii) Under which conditions is the cross ratio real, i.e. $T_c \in \mathbb{R}$? State this as a theorem. No proof is required.

iv) Use the result from i), ii) and the theorem in iii) to argue that the linear fractional transformation can never map a geometrical configuration into a circle or a line if this configuration was not a circle or a line in the first place.

Turn over ...

3) Consider the balance of the heat flow through a solid under steady state conditions, which means that we have no time dependence. Suppose that the solid extends infinitely long into one space direction. Under these circumstances the dependence of the temperature $T(x,y)$ on the two remaining space dimensions x, y is governed by the Laplace equation

$$\Delta T = \partial_x^2 T + \partial_y^2 T = 0.$$

Determine the temperature dependence between two infinitely high walls at positions $x = \pm\pi/2$, which are kept at temperature $T = 0$. The walls stand on a surface which is kept at temperature $T = 1/2$.

i) Formulate the above problem as a boundary values problem. Is this a Dirichlet or a Neumann boundary problem?

ii) Demonstrate that the two maps and the composition map

$$\tilde{w} = \tilde{f}(z) = \sin z, \quad w = \hat{f}(\tilde{w}) = \ln\left(\frac{\tilde{w}-1}{\tilde{w}+1}\right); \qquad f(z) = \hat{f} \circ \tilde{f}(z)$$

can be used to transform the boundary problem of i) into the easier problem of two parallel infinite plates. Solve the latter problem.

Hint: You may use the identities $\sin(x+iy) = \sin x \cosh y + i \cos x \sinh y$ and $\tanh^2(x/2) = (\cosh x - 1)/(\cosh x + 1)$.

iii) Use the solution from ii) and the map $f(z)$ to construct the solution

$$T(x,y) = \frac{1}{\pi} \arctan\left(\frac{\cos x}{\sinh y}\right)$$

for the boundary value problem in i).

Hint: You may use the identity $\tan 2x = 2\tan x/(1 - \tan^2 x)$.

iv) Verify explicitly that $T(x,y)$ solves the boundary problem in i), i.e. check explicitly the boundary conditions and the Laplace equation.

Turn over . . .

4) Consider a point particle of mass m, which is fixed on a spring with spring constant κ. When neglecting friction Newton's second law describes the motion of this particle as

$$m\ddot{x}(t) + \kappa x(t) = 0,$$

where x is the vertical displacement of the particle as a function of time t. Adding an external driving force $F(t)$, this system becomes the driven harmonic oscillator described by the equation

$$m\ddot{x}(t) + \kappa x(t) = F(t). \tag{dho}$$

Determine the vertical displacement of the particle as a function of time by following the instructions i)-iii):

i) For piecewise smooth function $u(x)$, with reasonable exponential growth, the Laplace transform of its nth derivative $\mathcal{L}u^{(n)}(x)$ may be expressed as

$$\mathcal{L}u^{(n)}(x) = x^n \mathcal{L}u(x) - \sum_{k=0}^{n-1} x^{n-k-1} u^{(k)}(0).$$

Prove this formula for $n = 1$ and $n = 2$. What is meant by exponential growth? State the convolution theorem for the Laplace transform.

ii) Use Laplace transforms with the properties of i) to solve equation (dho) for a generic external driving force, subject to the initial conditions $x(0) = \dot{x}(0) = 0$. Leave your answer in form of an integral representation on the real axis, that is not in form of the Bromwich integral.

Hint: You may use that

$$\mathcal{L}v(x) = \frac{\omega}{x^2 + \omega^2} \qquad \text{for } v(x) = \sin(\omega x), \lambda \in \mathbb{R}, x > 0.$$

iii) Specify now the function to be a kick at $t = 0$. Denoting by p the momentum transfer of this kick, the external force can be represented as

$$F(t) = p\delta(t).$$

Compute the integral left in ii).

Turn over ...

5) The Fourier transform $\mathcal{F}u(x) = \hat{u}(x)$ of a piecewise smooth and absolutely integrable function $u(x)$ on the real line is defined as

$$\mathcal{F}u(x) := \hat{u}(x) = \int_{-\infty}^{\infty} u(t)e^{-itx}dt.$$

i) Show that the Fourier transform for the scaled function $v(x) = u(\lambda x)$ with $\lambda \in \mathbb{R}^{+}$ is

$$\mathcal{F}v(x) = \mathcal{F}u(\lambda x) = \frac{1}{\lambda}\hat{u}(x/\lambda)$$

and the Fourier transform for the derivative $u'(x)$ of the function $u(x)$ is

$$\mathcal{F}u'(x) = ix\mathcal{F}u(x).$$

ii) Compute the Fourier transforms $\mathcal{F}u(x)$ of the function

$$u(x) = e^{-x^2}.$$

You may use the integral $\int_{-\infty}^{\infty} e^{-(t+ix/2)^2}dt = \sqrt{\pi}$.

iii) Use the relation between the Fourier transform and the Fourier transform of its derivative from i) and the result of ii) to compute the Fourier transforms $\mathcal{F}u(x)$ of the functions

$$u(x) = 2(2x^2 + x - 2)e^{-x^2}.$$

iv) Use Fourier transforms to solve the Fredholm integral equation for $\phi(x)$

$$\lambda e^{-x^2} = \int_{-\infty}^{\infty} e^{-(x-s)^2}\phi(s)ds,$$

where $\lambda \in \mathbb{R}$.

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Professor J. Billingham

Mathematical Methods II

Solutions Exam 07

1) i) **Definition:** *A map which preserves the angles between a pair of two intersecting lines is called a* conformal *map.* 1

Definition: *A function f of a complex variable z is said to be* analytic *in the domain $D \subset \mathbb{C}$ if its derivative exists for all $z \in D$. A function is said to be analytic in the point z_0 if there exists a neighbourhood around z_0 in which f in analytic.* 1

Theorem: *Any analytic function $f(z)$ defined on some domain $D \subset \mathbb{C}$ is conformal at the point $z_0 \in D$, if $f'(z_0) \neq 0$.* 1

ii) **Theorem:** *The linear fractional transformation $w = T(z)$ maps three distinct points z_1, z_2, z_3* uniquely *into three distinct points w_1, w_2, w_3. The map is determined by the equation* 1

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}.$$

Solving this for $z_1 = 1+i, z_2 = 3, z_3 = -2$, $w_1 = 1, w_2 = 0, w_3 = -1$ gives 5

$$w = T(z) = \frac{z-3}{(i-2)z + 1 + 2i}.$$

iii) First rotate the wedge region $\mathcal{W}$ by $-i\pi/8$

$$\hat{w} = \hat{f}(z) = ze^{-i\pi/8},$$

such that the new wedge region is

$$\mathcal{W}' = \{r, \theta : r \in \mathbb{R}^+, -\frac{\pi}{8} \leq \theta < \frac{\pi}{8}\}.$$

Next map this wedge to the entire right half plane by

$$\tilde{w} = \tilde{f}(\hat{w}) = \hat{w}^4.$$

Finally we map the right half plane to the unit circle

$$w = \check{f}(\tilde{w}) = \frac{\tilde{w}-1}{\tilde{w}+1}.$$

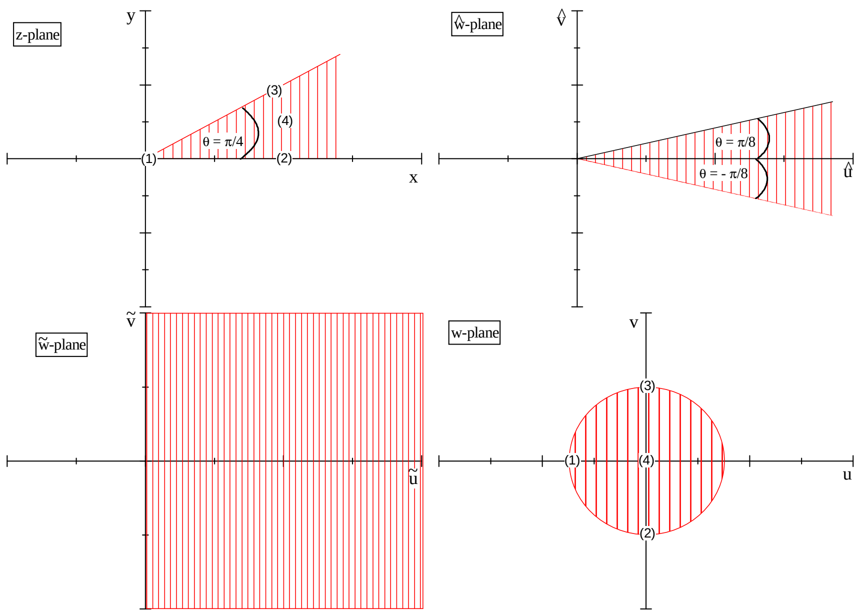
Thus the map which maps $\mathcal{W}$ onto the unit circle is 5

$$w = f(z) = \check{f} \circ \tilde{f} \circ \hat{f}(z) = \check{f} \circ \tilde{f}(ze^{-i\pi/8}) = \check{f}(-iz^4) = \frac{z^4-i}{z^4+i}.$$

Characteristic points: 1

$$1 : f(0) = -1, \quad 2 : f(1) = -i, \quad 3 : f(e^{i\pi/4}) = i, \quad 4 : f(e^{i\pi/8}) = 0.$$

Including some characteristic points into the figure:



3

Riemann mapping theorem: *Given a simply connected region $D \subset \mathbb{C}$ (i.e. D has no holes) which is not the entire plane and a point $z_0 \in D$. Then there exists an analytic function $f : z \mapsto w$ which maps D one-to-one onto the interior of the unit disk $|w| < 1$. The uniqueness of the map can be achieved with the normalization condition $f(z_0) = 0$ and $f'(z_0) > 0$.* 2

$\sum = 20$

2) i) **Definition:** *The transformation*

$$w = T(z) = \frac{az+b}{cz+d} \qquad \text{for } ad - bc \neq 0; a, b, c, d \in \mathbb{C}$$

is called the linear fractional transformation. 1

We can write

$$T^{c\neq 0}(z) = \frac{az+b}{c(z+d/c)} = \frac{a(z+d/c) - ad/c + b}{c(z+d/c)} = \frac{a}{c} + \frac{b - ad/c}{cz+d}.$$

Defining the translation map $f_T^{\Delta}(z) = z + \Delta$, the rotation map $f_R^{z_0}(z) = zz_0$ and the inversion map $f_I(z) = z^{-1}$ this can be brought into the form 3

$$T^{c\neq 0}(z) = f_T^{a/c} \circ f_R^{(bc-ad)/c} \circ f_I \circ f_T^{d} \circ f_R^{c}(z).$$

For $c = 0$ the linear fractional transformation $T(z)$ can be written as 1

$$T^{c=0}(z) = f_T^{b/d} \circ f_R^{a/d}(z).$$

ii) **Definition:** *The cross ratio T_c of the points (z_1, z_2, z_3, z_4) is the image of z_4 which maps the points (z_1, z_2, z_3) onto $(0,1,\infty)$* 1

$$T_c(z_4) = \frac{(z_4 - z_1)(z_2 - z_3)}{(z_4 - z_3)(z_2 - z_1)}.$$

- The difference $(z_i - z_j)$ is an invariant of f_T^Δ :

$$f_T^\Delta(z_i - z_j) = z_i + \Delta - z_j - \Delta = z_i - z_j$$

Since the cross ratio is a product and ratio of such differences it is also an invariant of the translation map

$$f_T^\Delta(T_c) = T_c.$$

- The ratio $(z_i - z_j)/(z_k - z_l)$ is an invariant of f_R^λ :

$$f_R^\lambda \left(\frac{z_i - z_j}{z_k - z_l} \right) = \frac{\lambda z_i - \lambda z_j}{\lambda z_k - \lambda z_l} = \frac{z_i - z_j}{z_k - z_l}$$

Since the cross ratio is a product of two such ratios it is also an invariant of the rotation map

$$f_R^\lambda(T_c) = T_c.$$

- T_c is an invariant of f_I :

$$f_I\,(T_c) = \frac{(z_4^{-1} - z_1^{-1})(z_2^{-1} - z_3^{-1})}{(z_4^{-1} - z_3^{-1})(z_2^{-1} - z_1^{-1})} = \frac{z_4 z_1 (z_4^{-1} - z_1^{-1})(z_2^{-1} - z_3^{-1}) z_2 z_3}{z_4 z_3 (z_4^{-1} - z_3^{-1})(z_2^{-1} - z_1^{-1}) z_2 z_1} = T_c$$

Since $T(z)$ is composed of f_T, f_R, f_I and T_c is an invariant of all individual transformations, it must also be an invariant of $T(z)$. 7

iii) **Theorem:** *The cross ratio T_c of the points (z_1, z_2, z_3, z_4) is real, i.e.*

$$\arg T_c(z_4) = \arg \frac{(z_4 - z_1)}{(z_4 - z_3)} - \arg \frac{(z_2 - z_1)}{(z_2 - z_3)} = 0 \text{ or } \pi,$$

if and only if the four points z_1, z_2, z_3, z_4 lie on a line or a circle. 2

iv) The Theorem in iii) states that the cross ratio is real if and only if (z_1, z_2, z_3, z_4) are on a circle or a line. We can map the points (z_1, z_2, z_3, z_4) into (w_1, w_2, w_3, w_4) by means of $T(z)$ and compute the cross ratio for (w_1, w_2, w_3, w_4). Once again the cross ratio is real if and only if (w_1, w_2, w_3, w_4) are on a circle or a line. Since T_c is 5
an invariant of $T(z)$ the cross ratios of (z_1, z_2, z_3, z_4) and (w_1, w_2, w_3, w_4) are the same. This means (w_1, w_2, w_3, w_4) can only be on a line or circle if (z_1, z_2, z_3, z_4) are on a circle or a line and vice versa.

$\sum = 20$

3) i) The boundary problem we have to solve is

$$\Delta T(x, y) = 0, \quad T(\pm\pi/2, y) = 0, \quad T(x, 0) = 1/2, \quad \text{for } |x| < \frac{\pi}{2}, y > 0.$$

This is a Dirichlet problem, since the values of the function $T(x, y)$ on the boundary are given. 2

ii) We compute the images of the boundaries. For the left wall we compute

$$f(z) = \hat{f} \circ \tilde{f}(-\frac{\pi}{2} + iy) = \ln\left(\frac{-\cosh y - 1}{-\cosh y + 1}\right) = \ln\left(\coth^2 y/2\right)$$

such that

$$f : -\frac{\pi}{2} \times [0, \infty) \mapsto [0, \infty) \times 0.$$

For the right wall we compute

$$f(z) = \hat{f} \circ \tilde{f}(\frac{\pi}{2} + iy) = \ln\left(\frac{\cosh y - 1}{\cosh y + 1}\right) = \ln\left(\tanh^2 y/2\right)$$

such that

$$f : \frac{\pi}{2} \times [0, \infty) \mapsto (-\infty, 0] \times 0.$$

For the ground we compute

$$f(z) = \hat{f} \circ \tilde{f}(\sin x) = \ln\left(\frac{\sin x - 1}{\sin x + 1}\right) = \ln\left|\frac{\sin x - 1}{\sin x + 1}\right| + i\pi$$

such that

$$f : [-\frac{\pi}{2}, \frac{\pi}{2}] \times 0 \mapsto (-\infty, \infty) \times i\pi.$$

Therefore we have mapped the original problem into the easier problem

$$\Delta T(x, y) = 0, \quad T(x, 0) = 0, \quad T(x, \pi) = 1/2, \quad \text{for } 0 < y < \pi.$$

This is directly solved by

$$T(x, y) = \frac{y}{2\pi}.$$

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iii) The solution to the Dirichlet problem in i) is therefore

$$T(x, y) = \frac{1}{2\pi} \operatorname{Im}(\hat{f} \circ \tilde{f}(z)) = \frac{1}{2\pi} \operatorname{Im}\left[\ln\left(\frac{\sin z - 1}{\sin z + 1}\right)\right].$$

We compute

$$w = \ln\left(\frac{\tilde{w} - 1}{\tilde{w} + 1}\right) = \ln\left|\frac{\tilde{w} - 1}{\tilde{w} + 1}\right| + i \arg\left(\frac{\tilde{w} - 1}{\tilde{w} + 1}\right) = u + iv,$$

such that

$$\begin{aligned} \operatorname{Im} w &= \arg\left(\frac{\tilde{w} - 1}{\tilde{w} + 1}\right) = \arg\left(\frac{\tilde{x} + i\tilde{y} - 1}{\tilde{x} + i\tilde{y} + 1}\right) = \arg\left[\frac{(\tilde{x} + i\tilde{y} - 1)(\tilde{x} + 1 - i\tilde{y})}{(\tilde{x} + 1 + i\tilde{y})(\tilde{x} + 1 - i\tilde{y})}\right], \\ &= \arg\left(\frac{\tilde{x}^2 + \tilde{y}^2 - 1 + i2\tilde{y})}{(\tilde{x} + 1)^2 + \tilde{y}^2}\right) = \arctan\left(\frac{2\tilde{y}}{\tilde{x}^2 + \tilde{y}^2 - 1}\right). \end{aligned}$$

Next we need to express $\tilde{x}, \tilde{y}$ in terms of x, y. We have

$$\tilde{w} = \sin(z) = \sin(x + iy) = \sin x \cosh y + i \cos x \sinh y = \tilde{x} + i\tilde{y}$$

and therefore

$$\begin{aligned}\operatorname{Im} w &= \arctan\left(\frac{2\cos x \sinh y}{\sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y - 1}\right)\\ &= \arctan\left(\frac{2\cos x \sinh y}{\sinh^2 y - \cos^2 x}\right) = \arctan\left(\frac{2\cos x/\sinh y}{1-\cos^2 x/\sinh^2 y}\right).\end{aligned}$$

Introducing now the auxiliary variable $\tan\gamma = \cos x/\sinh y$ we can use the identity $\tan 2\gamma = 2\tan\gamma/(1-\tan^2\gamma)$ and obtain

$$\operatorname{Im} w = \arctan(\tan 2\gamma) = 2\gamma = 2\arctan\left(\frac{\cos x}{\sinh y}\right).$$

This means the solution to the boundary Dirichlet problem i) is

$$T(x,y) = \frac{1}{\pi}\arctan\left(\frac{\cos x}{\sinh y}\right).$$

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iv) We may easily check that the boundary condition are indeed satisfied

$$\begin{aligned}T(\pm\pi/2, y) &= \frac{1}{\pi}\arctan\left(\frac{\cos(\pm\pi/2)}{\sinh y}\right) = \frac{1}{\pi}\arctan 0 = 0,\\ T(x,0) &= \frac{1}{\pi}\arctan\left(\frac{\cos x}{\sinh 0}\right) = \frac{1}{\pi}\lim_{x\to\infty}\arctan(x) = 1/2.\end{aligned}$$

The Laplace equation is verified by

$$\begin{aligned}\partial_x \arctan\left(\frac{\cos x}{\sinh y}\right) &= -\frac{\sin x \sinh y}{\sinh^2 y + \cos^2 x}\\ \partial_x^2 \arctan\left(\frac{\cos x}{\sinh y}\right) &= -\frac{\cos x \sinh y}{\sinh^2 y + \cos^2 x} - \frac{2\cos x \sinh y \sin^2 x}{(\sinh^2 y + \cos^2 x)^2}\\ \partial_y \arctan\left(\frac{\cos x}{\sinh y}\right) &= -\frac{\cos x \cosh y}{\sinh^2 y + \cos^2 x}\\ \partial_y^2 \arctan\left(\frac{\cos x}{\sinh y}\right) &= -\frac{\cos x \sinh y}{\sinh^2 y + \cos^2 x} + \frac{2\cos x \sinh y \cosh^2 y}{(\sinh^2 y + \cos^2 x)^2}.\end{aligned}$$

Therefore 6

$$\begin{aligned}\Delta \arctan\left(\frac{\cos x}{\sinh y}\right) &= -\frac{2\cos x \sinh y}{\sinh^2 y + \cos^2 x} + \frac{2\cos x \sinh y(\cosh^2 y - \sin^2 x)}{(\sinh^2 y + \cos^2 x)^2}\\ &= \frac{2\cos x \sinh y(-\sinh^2 y - \cos^2 x + \cosh^2 y - \sin^2 x)}{(\sinh^2 y + \cos^2 x)^2}\\ &= 0.\end{aligned}$$

$\sum = 20$

4) i) For $n = 1$ we have

$$\mathcal{L}u'(x) = \int_0^\infty u'(t)e^{-tx}dt = u(t)e^{-tx}\big|_0^\infty + x\int_0^\infty u(t)e^{-tx}dt = -u(0) + x\mathcal{L}u(x)$$

For $n = 2$ we have

$$\begin{aligned}\mathcal{L}u''(x) &= \int_0^\infty u''(t)e^{-tx}dt = u'(t)e^{-tx}\big|_0^\infty + x\int_0^\infty u'(t)e^{-tx}dt = -u'(0) + x\mathcal{L}u'(x) \\ &= -u'(0) - xu(0) + x^2\mathcal{L}u(x)\end{aligned}$$

2

Definition: *The function $u(x)$ is said to have exponential growth α if there exists a constant μ such that*

$$|u(x)| \leq \mu e^{\alpha x} \qquad \text{for } x > 0, \text{ with } \alpha, \mu \in \mathbb{R}.$$

1

Convolution theorem: *The Laplace transform of the convolution of the two functions u and v, i.e. $u \star v(x)$ equals the product of the Laplace transforms these functions*

$$\mathcal{L}(u \star v)(x) = (\mathcal{L}u)(x)(\mathcal{L}v)(x).$$

1

ii) Acting with the Laplace operator $\mathcal{L}$ on

$$m\ddot{x}(t) + \kappa x(t) = F(t)$$

gives

$$m\mathcal{L}\ddot{x}(t) + \kappa\mathcal{L}x(t) = \mathcal{L}F(t).$$

Using the two formulae from i) and the initial conditions $x(0) = \dot{x}(0) = 0$

$$\begin{aligned}\mathcal{L}\ddot{x}(t) &= t^2\mathcal{L}x(t) - tx(0) - \dot{x}(0) \qquad \Rightarrow \qquad \mathcal{L}\ddot{x}(t) = t^2\mathcal{L}x(t) \\ \mathcal{L}\dot{x}(t) &= t\mathcal{L}x(t) - x(0) \qquad \Rightarrow \qquad \mathcal{L}\dot{x}(t) = t\mathcal{L}x(t).\end{aligned}$$

Therefore we can rewrite the above equation as

$$mt^2\mathcal{L}x(t) + \kappa\mathcal{L}x(t) = \mathcal{L}F(t).$$

Solving this for $\mathcal{L}x(t)$

$$\mathcal{L}x(t) = \frac{\mathcal{L}F(t)}{mt^2 + \kappa} = \frac{\mathcal{L}F(t)}{m(t^2 + \kappa/m)} = \frac{\mathcal{L}F(t)}{m}\frac{1}{t^2 + \omega^2}.$$

Here we abbreviated $\omega^2 = \kappa/m$. Using now the hint

$$\mathcal{L}v(t) = \frac{\omega}{t^2 + \omega^2} \qquad \text{for} \qquad v(t) = \sin\omega t,$$

we can rewrite the above equation as

$$\mathcal{L}x(t) = \frac{1}{m\omega}\mathcal{L}F(t)\mathcal{L}v(t) = \frac{1}{m\omega}\mathcal{L}(F{*}v)(t),$$

where we used the convolution theorem in the last equality. Acting now with $\mathcal{L}^{-1}$ on this equation yields the final asnwer for $x(t)$ in form of an integral representation

$$x(t) = \frac{1}{m\omega}F{*}v(t) = \frac{1}{m\omega}\int_0^\infty ds\ F(t-s)\sin\omega s.$$

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iii) Specifying now the function to be a kick at $t = 0$ we have to compute 2

$$x(t) = \frac{p}{m\omega}\int_0^\infty ds\ \delta(t-s)\sin\omega s = \frac{p}{m\omega}\ \sin\omega t.$$

$\sum = 20$

5) i) The Fourier transform for the scaled function $v(x) = u(\lambda x)$ with $\lambda \in \mathbb{R}^+$ is 2

$$\mathcal{F}v(x) = \mathcal{F}u(\lambda x) = \int_{-\infty}^{\infty} u(\lambda t)e^{-itx}dt = \frac{1}{\lambda}\int_{-\infty}^{\infty} u(s)e^{-isx/\lambda}ds = \frac{1}{\lambda}\hat{u}(x/\lambda).$$

The Fourier transform for the derivative $u'(x)$ of the function $u(x)$ is 2

$$\mathcal{F}u'(x) = \int_{-\infty}^{\infty} u'(t)e^{-itx}dt = u(t)e^{-itx}\big|_{-\infty}^{\infty} + ix\int_{-\infty}^{\infty} u(t)e^{-itx}dt = ix\mathcal{F}u(x).$$

ii) We compute

$$\begin{aligned}\mathcal{F}u(x) &= \int_{-\infty}^{\infty} e^{-t^2}e^{-itx}dt = \int_{-\infty}^{\infty} e^{-(t+ix/2)^2}e^{-x^2/4}dt \\ &= e^{-x^2/4}\int_{-\infty}^{\infty} e^{-(t+ix/2)^2}dt = \sqrt{\pi}e^{-x^2/4}.\end{aligned}$$

We used the integral $\int_{-\infty}^{\infty} e^{-(t+ix/2)^2}dt = \sqrt{\pi}$. 4

iii) Define $v(x) = e^{-x^2}$. Then $v'(x) = -2xe^{-x^2}$and $v''(x) = (4x^2-2)e^{-x^2}$. Therefore

$$\begin{aligned}u(x) &= v''(x) - v'(x) - 2v(x) \\ &= 4x^2e^{-x^2} + 2xe^{-x^2} - 4e^{-x^2},\end{aligned}$$

such that

$$\begin{aligned}\mathcal{F}u(x) &= \mathcal{F}v''(x) - \mathcal{F}v'(x) - 2\mathcal{F}v(x) \\ &= ix\mathcal{F}v'(x) - \mathcal{F}v'(x) - 2\mathcal{F}v(x) \\ &= (ix-1)ix\mathcal{F}v(x) - 2\mathcal{F}v(x) \\ &= -\sqrt{\pi}(x^2+ix+2)e^{-x^2/4}.\end{aligned}$$

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iv) Introducing the function $v(x) = e^{-x^2}$, we can rewrite

$$\lambda e^{-x^2} = \int_{-\infty}^{\infty} e^{-(x-s)^2} \phi(s) ds$$

as

$$\lambda v(x) = v * \phi(x)$$

Acting on this equation with the Fourier operator $\mathcal{F}$ gives

$$\lambda \mathcal{F} v(x) = \mathcal{F}(v * \phi)(x) = \mathcal{F} v(x) \mathcal{F} \phi(x).$$

Therefore we obtain

$$\lambda = \mathcal{F} \phi(x),$$

such that

$$\phi(x) = \lambda \mathcal{F}^{-1} 1(x) = \lambda \delta(x).$$

6

$\sum = 20$