MA 3605

CITY UNIVERSITY

London

BSc Degrees in Mathematical Science
Mathematical Science with Statistics
Mathematical Science with Computer Science
Mathematical Science with Finance and Economics
MMath Degrees in Mathematical Science

Part III Examination

Mathematical Methods

May 2008

Time allowed: 2 hours

Full marks may be obtained for correct answers to THREE of the FIVE questions.

If more than THREE questions are answered, the best THREE marks will be credited.

Turn over ...

Each question carries 25 marks.

1) i) (5 marks) For an analytic function $f(x, y) = u(x, y)+iv(x, y)$ on a domain D state the Cauchy-Riemann equations. Use these equations to construct $f(x, y)$ with given harmonic function $u(x, y) = \cosh x \cos y$.

ii) (9 marks) Provide a definition for a linear fractional transformation $T(z)$. Determine the linear fractional transformation $w = f(z)$, which maps the points $z_1 = i, z_2 = 0, z_3 = 1$ in the z-plane onto $w_1 \to \infty, w_2 = 3i, w_3 = -1$ in the w-plane. Is this map unique?

iii) (7 marks) Show that $T(z)$ can be written as

$$T(z) = \frac{a}{c} + \frac{b - ad/c}{cz + d} \quad \text{for } c \neq 0, \qquad \text{with } a, b, c, d \in \mathbb{R}$$

such that it may be understood as a composition of more elementary transformations, i.e. what is usually referred to as a translation f_T^Δ, a rotation f_R^λ and the inversion map f_I. Express $T(z)$ in terms of these maps. Provide also the decomposition of $T(z)$ into these type of maps for the case $c = 0$.

iv) (1 mark) Given are two linear fractional transformations $T_1(z)$ and $T_2(z)$. What type of map is obtained by the composition of these two maps $T_1 \circ T_2(z)$?

v) (3 marks) For the linear fractional transformations

$$T_1(z) = \frac{z - 1}{z - i} \qquad \text{and} \qquad T_2(z) = \frac{z - i}{z - 1}$$

compute $f_I \circ T_1 \circ T_2(z)$ and $f_R^{1-i} \circ f_T^{-1} \circ T_1(z)$.

2) i) (3 marks) State the Schwarz-Christoffel theorem.

ii) (19 marks) Determine the Schwarz-Christoffel transformation, which maps the upper half plane onto an isosceles right triangle. Map the points $x_1 = -1$, $x_2 = 1$ and $x_3 \to \infty$ to $w_1 = ia$, $w_2 = 0$ and $w_3 = a$. Express your result in terms of the quantity

$$\kappa = \int_{-1}^{1} d\hat{z} \frac{1}{(1 + \hat{z})^{3/4}(1 - \hat{z})^{1/2}} = \frac{\sqrt{\pi}}{2^{1/4}} \frac{\Gamma(1/4)}{\Gamma(3/4)} \approx 4.40976.$$

iii) (3 marks) Draw the corresponding w-plane.

3) i) (5 marks) Define the exponential growth of a function $f(x)$ and subsequently define the Laplace tranform $\mathcal{L}f(x)$ for the function $f(x)$.

ii) (20 marks) Find the inverse Laplace transform $\mathcal{L}^{-1}v(x)$ for the function

$$v(x) = \frac{\omega}{x^2 - \omega^2} \qquad \text{for } \omega \in \mathbb{R},$$

by computing the Bromwich integral

$$\mathcal{L}^{-1}v(x) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} v(t) e^{tx} dt \qquad \text{for } \gamma > \alpha,$$

where α is the exponential growth of $\mathcal{L}^{-1}v(x)$.

4) Consider two infinite cylinders placed non-coaxially with $|z| = 1$ and $|z - x_0| = x_0$. The cylinders are at constant potential $\phi_1 = 0V$ at $|z| = 1$ and $\phi_0 = 220V$ at $|z - x_0| = x_0$. The value of the center of the smaller cylinder and its radius is taken to be $x_0 = 2/5$. Find the potential in the xy-plane between the two cylinders.

i) (2 marks) Draw the corresponding figure.

ii) (4 marks) Verify first that the potential for two infinitely long cylinders with radii r_0 and $r = 1$ at potentials ϕ_0, ϕ_1 is given by

$$\phi(r) = (\phi_0 - \phi_1) \frac{\ln r}{\ln r_0} + \phi_1.$$

Hint: The Laplace equation in polar coordinates is

$$\Delta\phi = \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \vartheta^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} = 0.$$

iii) (16 marks) Verify that the map

$$w = f(z) = \frac{z - c}{cz - 1} \qquad \text{with } c \in \mathbb{R}$$

leaves the circle with the radius $|z| = 1$ invariant, that is the image is $|w| = 1$. Fix the constant c, such that the circle $|z - x_0| = x_0$ is mapped onto $|w| = r_0$.

iv) (3 marks) Use the results from ii) and iii) to compute the potential for the non-coaxial cylinders.

Turn over ...

5) The Fourier transform $\mathcal{F}u(x) = \hat{u}(x)$ of a piecewise smooth and absolutely integrable function $u(x)$ on the real line is defined as

$$\mathcal{F}u(x) := \hat{u}(x) = \int_{-\infty}^{\infty} u(t)e^{-itx}dt.$$

i) (5 marks) Define the convolution $u \star v(x)$ of two functions $v(x)$ and $u(x)$. Then show that the Fourier transform of the convolution of two functions $v(x)$ and $u(x)$ equals the product of the Fourier transforms of $v(x)$ and $u(x)$.

ii) (3 marks) Compute the Fourier transforms $\mathcal{F}u(x)$ of the function

$$u(x) = e^{-x^2}.$$

You may use the integral $\int_{-\infty}^{\infty} e^{-(t+ix/2)^2}dt = \sqrt{\pi}$.

iii) (9 marks) Use the relation between the Fourier transform and the Fourier transform of its derivative $\mathcal{F}u'(x) = ix\mathcal{F}u(x)$ and the result of ii) to compute the Fourier transform $\mathcal{F}u(x)$ of the function

$$u(x) = 2(8x^2 + 3x - 3)e^{-x^2}.$$

iv) (8 marks) Use Fourier transforms to solve the Fredholm integral equation for $\phi(x)$

$$e^{-x^2} = \kappa \int_{-\infty}^{\infty} e^{-(x-s/\lambda)^2}\phi(s/\lambda)ds,$$

where $\lambda, \kappa \in \mathbb{R}$.

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Professor E. Corrigan

Mathematical Methods II

Solutions Exam 08

1) i) For an analytic function $f(x,y) = u(x,y) + iv(x,y)$ on a domain D the Cauchy-Riemann equations are $\boxed{\sum = 25}$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

$\boxed{1}$

Therefore with $u(x,y) = \cosh x \cos y$ follows

$$\begin{aligned} \frac{\partial u}{\partial x} &= \cos y \sinh x = \frac{\partial v}{\partial y} \Rightarrow v = \sinh x \int \cos y dy = \sinh x \sin y + \sinh x g(x) \\ \frac{\partial u}{\partial y} &= -\sin y \cosh x = -\frac{\partial v}{\partial x} \Rightarrow v = \sin y \int \cosh x dx = \sinh x \sin y + \sinh y h(y) \end{aligned}$$

such that $g(x) = h(x) = 0$ and $f(x,y) = \cosh x \cos y + i \sinh x \sin y$. $\boxed{4}$

ii) **Definition:** The transformation

$$w = T(z) = \frac{az+b}{cz+d} \qquad \text{for } ad - bc \neq 0; a,b,c,d \in \mathbb{C}$$

is called the *linear fractional transformation*. $\boxed{1}$

Theorem: The linear fractional transformation $w = T(z)$ maps three distinct points z_1, z_2, z_3 uniquely into three distinct points w_1, w_2, w_3. The map is determined by the equation

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}.$$

$\boxed{3}$

Solving this for $z_1 = i, z_2 = 0, z_3 = 1$, $w_1 \to \infty, w_2 = 3i, w_3 = -1$ gives

$$\frac{(w_2+1)}{(w+1)} = \frac{(z-i)(0-1)}{(z-1)(0-i)} \Rightarrow w = T(z) = \frac{(i-4)z+3}{z-i}.$$

$\boxed{5}$

iii) We can write

$$T^{c\neq 0}(z) = \frac{az+b}{c(z+d/c)} = \frac{a(z+d/c) - ad/c + b}{c(z+d/c)} = \frac{a}{c} + \frac{b - ad/c}{cz+d}.$$

Defining the translation map $f_T^{\Delta}(z) = z + \Delta$, the rotation map $f_R^{z_0}(z) = zz_0$ and the inversion map $f_I(z) = z^{-1}$ this can be brought into the form $\boxed{5}$

$$T^{c\neq 0}(z) = f_T^{a/c} \circ f_R^{(bc-ad)/c} \circ f_I \circ f_T^{d} \circ f_R^{c}(z).$$

For $c = 0$ the linear fractional transformation $T(z)$ can be written as $\boxed{2}$

$$T^{c=0}(z) = f_T^{b/d} \circ f_R^{a/d}(z).$$

iv) The set of linear fractional transformations constitutes a group, such that $T_1(z) \circ T_2(z)$ is also a linear fractional transformation (by the closure of the group). $\boxed{1}$

v) We have

$$T_1(z) = \frac{z-1}{z-i} \qquad \text{and} \qquad T_2(z) = \frac{z-i}{z-1}$$

$$\begin{aligned} f_I \circ T_1 \circ T_2(z) &= f_I\left(\frac{\frac{z-i}{z-1} - 1}{\frac{z-i}{z-1} - i}\right) = f_I\left(\frac{1}{z}\right) = z \\ f_R^{1-i} \circ f_T^{-1} \circ T_1(z) &= (1-i)\left(\frac{z-1}{z-i} - 1\right) = \frac{2i}{z-i}. \end{aligned}$$

$\boxed{3}$

2) i) Schwarz-Christoffel theorem: Given an n-sided polygon with vertices w_i and exterior angles $\theta_i = \mu_i \pi$ for $1 \leq i \leq n$. Then there exist always n real numbers x_i for $1 \leq i \leq n$ together with a complex constant $c \in C$ and an analytic function $f : z \mapsto w$ whose derivative is given by $\boxed{\sum = 25}$

$$f'(z) = c \prod_{i=1}^{n-1} (z - x_i)^{-\mu_i} \qquad c \in \mathbb{C}, -1 < \mu_i < 1,$$

which maps the upper half plane one-to-one onto the interior of the polygon. The points are mapped as $w_i = f(x_i)$ for $1 \leq i \leq n-1$ and $w_n = \lim_{x \to \pm\infty} f(x)$. $\boxed{3}$

ii) The exterior angles at w_1, w_2, w_3 are $3\pi/4, \pi/2$ and $3\pi/4$. According to the Schwarz-Christoffel theorem the map is therefore given as

$$f'(z) = c(z+1)^{-3/4}(z-1)^{-1/2}.$$

Therfore

$$f(z) = c \int_1^z d\hat{z}(\hat{z}+1)^{-3/4}(\hat{z}-1)^{-1/2} + \tilde{c}.$$

We have

$$f(1) = w_2 = 0 \qquad \Rightarrow \qquad \tilde{c} = 0.$$

We also have

$$\begin{aligned} f(-1) &= c \int_1^{-1} d\hat{z}(\hat{z}+1)^{-3/4}(\hat{z}-1)^{-1/2} \\ &= c(-1) \int_{-1}^{1} d\hat{z}(\hat{z}+1)^{-3/4}(1-\hat{z})^{-1/2}(-1)^{-1/2} \\ &= ic\kappa = w_1 = ia \end{aligned}$$

and therefore

$$c = a/\kappa.$$

This means the transformation which maps the upper half plane onto the specified isosceles right triangle is

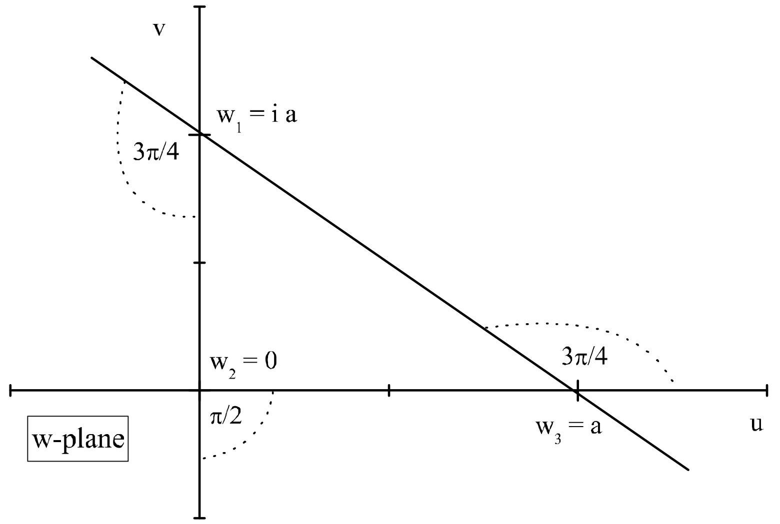
$$f(z) = \frac{a}{\kappa} \int_1^z d\hat{z}(\hat{z}+1)^{-3/4}(\hat{z}-1)^{-1/2}.$$

The Schwarz-Christoffel theorem guarantees the existence of this map. $\boxed{19}$
(In addition one may verify that

$$w_3 = \frac{a}{\kappa}\int_1^\infty d\hat{z}(\hat{z}+1)^{-3/4}(\hat{z}-1)^{-1/2} = a,$$

but this was not asked.)

iii) The corresponding figure is



$\boxed{3}$

3) i) **Definition:** The function $f(x)$ is said to have <u>*exponential growth*</u> α if there exists a constant μ such that $\boxed{\sum = 25}$

$$|f(x)| \leq \mu e^{\alpha x} \qquad \text{for } x > 0, \text{ with } \alpha, \mu \in \mathbb{R}.$$

$\boxed{2}$

Definition: The <u>*Laplace transform*</u> $\mathcal{L}u(x)$ of a piecewise smooth function $f(x)$ with exponential growth α is defined as

$$\mathcal{L}f(x) := \int_0^\infty f(t)e^{-tx}dt \qquad \text{for } x > \alpha.$$

$\boxed{3}$

ii) Compute

$$\mathcal{L}^{-1}v(x) = \frac{1}{2\pi i}\int\limits_{\gamma-i\infty}^{\gamma+i\infty} \frac{\omega}{t^2-\omega^2}e^{tx}dt.$$

Parameterize $z = \varepsilon + re^{i\theta}$ and compute

$$\frac{1}{2\pi i}\oint\limits_\Gamma \frac{\omega}{z^2-\omega^2}e^{zx}dz = \frac{2\pi i}{2\pi i}\operatorname*{Res}_{z_0=\pm\omega}\frac{\omega}{(z-\omega)(z+\omega)}e^{zx} = \frac{1}{2}e^{x\omega} - \frac{1}{2}e^{-x\omega} = \sinh\omega x$$

In order to show that $\boxed{8}$

$$\int\limits_{\gamma-i\infty}^{\gamma+i\infty} \frac{t}{t^2-\omega^2} e^{tx} dt = \oint\limits_{\Gamma} \frac{z}{z^2-\omega^2} e^{zx} dz$$

we have to guarantee that the integral over, say γ, parameterized by $re^{i\theta}$ for θ from $\pi/2$ to $3\pi/2$ vanishes as $r \to \infty$. Compute

$$\left| \oint\limits_{\gamma} \frac{\omega}{z^2-\omega^2} e^{zx} dz \right| = \left| \omega e^{\varepsilon x} \int\limits_{\pi/2}^{3\pi/2} \frac{re^{i\theta}}{(\varepsilon + re^{i\theta})^2 - \omega^2} e^{re^{i\theta}x} d\theta \right|$$

With

$$\begin{aligned} \left|re^{i\theta}\right| &= r \\ \left|e^{re^{i\theta}x}\right| &= e^{rx\cos\theta} \leq 1 \qquad\qquad \text{for } \frac{\pi}{2} \leq \theta \leq \frac{3}{2}\pi \\ \left|(\varepsilon + re^{i\theta})^2 - \omega^2\right| &> r^2 - \omega^2 \end{aligned}$$

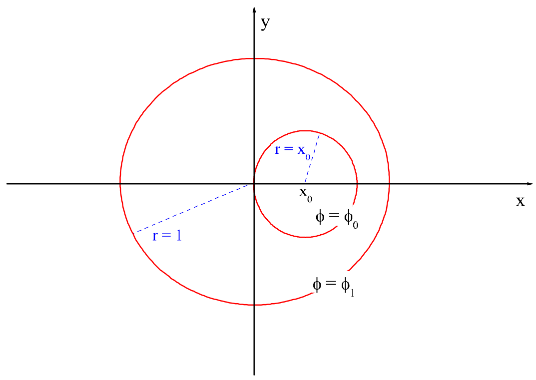
follows

$$\left| e^{\varepsilon x} \int\limits_{\pi/2}^{3\pi/2} \frac{re^{i\theta}}{(\varepsilon + re^{i\theta})^2 - \omega^2} e^{re^{i\theta}x} d\theta \right| < e^{\varepsilon x} \frac{r}{r^2-\omega^2} \to 0 \;\; \text{for } r \to \infty.$$

$$\Rightarrow \mathcal{L}^{-1} v(x) = \sinh \omega x$$

$\boxed{12}$

4) i) Non-coaxial cylinders: $\boxed{\sum = 25}$



$\boxed{2}$

ii) As there is no ϑ dependence we have $\partial\phi/\partial\vartheta = 0$, such that

$$\Delta\phi = \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\vartheta^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} = 0$$

reduces to

$$\Delta\phi = 0 \qquad \Leftrightarrow \qquad r\frac{\partial^2\phi}{\partial r^2} + \frac{\partial\phi}{\partial r} = 0.$$

For $\phi(r) = (\phi_0 - \phi_1)\frac{\ln r}{\ln r_0} + \phi_1$ compute

$$\left.\begin{array}{rcl} \frac{\partial\phi}{\partial r} &=& \frac{(\phi_0-\phi_1)}{\ln r_0}\frac{1}{r} \\ \frac{\partial^2\phi}{\partial r^2} &=& -\frac{(\phi_0-\phi_1)}{\ln r_0}\frac{1}{r^2} \end{array}\right\} \Rightarrow \Delta\phi = 0.$$

$\boxed{4}$

iii) Use the fact that the map is a linear fractional transformation and that the latter maps three points uniquely onto three points. Take therefore three distinct points on the unit circle in the z-plane $z_1 = 1, z_2 = -1, z_3 = i$ and compute

$$\begin{aligned} f(1) &= -1 \in |w| = 1 \\ f(-1) &= 1 \in |w| = 1 \\ f(i) &= \frac{i-c}{ic-1} = \frac{(i-c)(-ic-1)}{(ic-1)(-ic-1)} = \frac{2c}{1+c^2} + i\frac{(c^2-1)}{1+c^2} \end{aligned}$$

We see that $[\operatorname{Re} f(i)]^2 + [\operatorname{Im} f(i)]^2 = 1$, such that $f(i) \in |w| = 1$. $\boxed{4}$

From the geometry we have

$$\begin{aligned} f(0) &= r_0 \qquad \Rightarrow \qquad c = r_0 \\ f(2x_0) &= -r_0 \qquad \Rightarrow \qquad \frac{2x_0 - r_0}{2x_0 r_0 - 1} = -r_0 \end{aligned}$$

Combining these equations gives a quadratic equation in r_0

$$2x_0 - r_0 + 2x_0 r_0^2 - r_0 = 0 \quad \Rightarrow \quad r_0^{(1/2)} = \frac{1}{2x_0} \pm \frac{1}{2x_0}\sqrt{1 - 4x_0^2}.$$

Taking now $x_0 = 2/5$

$$r_0^{(1/2)} = \frac{5}{4} \pm \frac{5}{4}\sqrt{1 - 16/25} = \frac{5}{4} \pm \frac{5}{4}\frac{3}{5} \Rightarrow r_0^{(1)} = 2, r_0^{(2)} = \frac{1}{2}$$

Since the cylinder at radius r_0 has to be in the inside of the cylinder with radius $r = 1$, i.e. $r_0 < 1$, we discard the solution $r_0^{(1)}$. Therefore we the resulting conformal map is

$$f(z) = \frac{z - 1/2}{z/2 - 1} = \frac{2z - 1}{z - 2}.$$

$\boxed{12}$

iv)

$$\phi(r) = \phi_{\text{coaxial}}(f(r)) = (\phi_0 - \phi_1)\frac{\ln(f(r))}{\ln r_0} + \phi_1 = -220\frac{\ln\left|\frac{2z-1}{z-2}\right|}{\ln 2}V.$$

$\boxed{3}$

5) i) Definition: The <u>*convolution*</u> of two functions $u(x)$ and $v(x)$ is defined as $\boxed{\sum = 25}$

$$u \star v(x) = \int_{-\infty}^{\infty} u(t)v(x-t)dt.$$

From the definition follows $\boxed{1}$

$$\begin{aligned}
\mathcal{F}(u\star v)(x) &= \int_{-\infty}^{\infty} (u\star v)(t)e^{-itx}dt = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} ds\ u(s)v(t-s)e^{-itx} \\
&= \int_{-\infty}^{\infty} ds\ u(s)\left(\int_{-\infty}^{\infty} dt\ v(t-s)e^{-itx}e^{isx}\right) e^{-isx} \\
&= \int_{-\infty}^{\infty} ds\ u(s)e^{-isx}\left(\int_{-\infty}^{\infty} dt\ v(t)e^{-itx}\right) \\
&= \mathcal{F}(u)\mathcal{F}(v).
\end{aligned}$$

$\boxed{4}$

ii) We compute

$$\begin{aligned}
\mathcal{F}u(x) &= \int_{-\infty}^{\infty} e^{-t^2}e^{-itx}dt = \int_{-\infty}^{\infty} e^{-(t+ix/2)^2}e^{-x^2/4}dt \\
&= e^{-x^2/4}\int_{-\infty}^{\infty} e^{-(t+ix/2)^2}dt = \sqrt{\pi}e^{-x^2/4}.
\end{aligned}$$

We used the integral $\int_{-\infty}^{\infty} e^{-(t+ix/2)^2}dt = \sqrt{\pi}$. $\boxed{3}$

iii) Define $v(x) = e^{-x^2}$. Then $v'(x) = -2xe^{-x^2}$and $v''(x) = (4x^2-2)e^{-x^2}$. Therefore

$$\begin{aligned}
u(x) &= \alpha v(x) + \beta v'(x) + \gamma v''(x) \\
&= \left[\alpha - 2\beta x + \gamma(4x^2-2)\right] e^{-x^2} \\
&= (16x^2+6x-6)e^{-x^2}
\end{aligned}$$

$\Rightarrow \alpha - 2\gamma = -6,\ 4\gamma = 16,\ -2\beta = 6 \Rightarrow \gamma = 4,\ \beta = -3,\ \alpha = 2 \Rightarrow$

$$u(x) = 2v(x) - 3v'(x) + 4v''(x)$$

Therefore

$$\begin{aligned}
\mathcal{F}u(x) &= 4\mathcal{F}v''(x) - 3\mathcal{F}v'(x) + 2\mathcal{F}v(x) \\
&= 4ix\mathcal{F}v'(x) - 3ix\mathcal{F}v(x) + 2\mathcal{F}v(x) \\
&= 4(ix)^2\mathcal{F}v(x) - 3ix\mathcal{F}v(x) + 2\mathcal{F}v(x) \\
&= \sqrt{\pi}(-4x^2 - 3ix + 2)e^{-x^2/4}.
\end{aligned}$$

$\boxed{9}$

iv) First scale $s \to \lambda s$, such that the equation becomes

$$e^{-x^2} = \lambda\kappa \int_{-\infty}^{\infty} e^{-(x-s)^2} \phi(s) ds$$

Introducing the function $v(x) = e^{-x^2}$, this can be rewritten as

$$v(x) = \lambda\kappa v * \phi(x)$$

Acting on this equation with the Fourier operator $\mathcal{F}$ gives

$$\mathcal{F}v(x) = \mathcal{F}(\lambda\kappa v * \phi)(x) = \lambda\kappa\mathcal{F}(v * \phi)(x) = \lambda\kappa\mathcal{F}v(x)\mathcal{F}\phi(x).$$

Therefore we obtain

$$\frac{1}{\lambda\kappa} = \mathcal{F}\phi(x),$$

such that

$$\phi(x) = \mathcal{F}^{-1}\frac{1}{\lambda\kappa}(x) = \frac{1}{\lambda\kappa}\mathcal{F}^{-1}1(x) = \frac{1}{\lambda\kappa}\delta(x).$$

$\boxed{8}$