MA 3605

CITY UNIVERSITY

London

BSc Degrees in Mathematical Science
Mathematical Science with Statistics
Mathematical Science with Computer Science
Mathematical Science with Finance and Economics
MMath Degrees in Mathematical Science

PART III EXAMINATION

Mathematical Methods

May 2009

Time allowed: 2 hours

Full marks may be obtained for correct answers to THREE of the FIVE questions.

If more than THREE questions are answered, the best THREE marks will be credited.

Turn over ...

Each question carries 25 marks.

1) (*i*) (5 marks) State the definition for the cross ratio and a theorem involving it, which may be used to decide whether four points in the complex plane lie on a line or a circle. Compute the cross ratio $(3, -3, 3i, z_4)$ and use the theorem to decide whether the points $z_1 = 1 + i\sqrt{8}$ and $z_2 = 3 + i$ lie on the circle $|z| = 3$.

(*ii*) (11 marks) The linear fractional transformation is defined as

$$w = T(z) = \frac{az+b}{cz+d} \qquad \text{for } ad - bc \neq 0; a, b, c, d \in \mathbb{C}.$$

For which choices of a, b, c, d does this reduce to a translation, a rotation and the inversion map? Why is the case $ad - bc = 0$ excluded? Determine the linear fractional transformation $T(z)$, which maps the points $z_1 = i, z_2 = 0, z_3 = 1$ in the z-plane onto $w_1 = -1, w_2 \to \infty, w_3 = -i$ in the w-plane. Is this map unique?

(*iii*) (9 marks) Using the parameterization of all possible circles and lines in the z-plane

$$\alpha(x^2 + y^2) + \beta x + \gamma y + \delta = 0 \qquad \text{for } \alpha, \beta, \gamma, \delta \in \mathbb{R},$$

prove for the inversion map obtained in (*ii*) that it always maps circles and lines into circles and lines. For which values of $\alpha, \beta, \gamma, \delta$ are circles mapped into circles, circles mapped into lines, lines mapped into circles and lines mapped into lines?

2) (*i*) (3 marks) State the Schwarz-Christoffel theorem.

(*ii*) (11 marks) Determine the Schwarz-Christoffel transformation $f(z)$, which maps the upper half plane onto an equilateral triangle. Map the points $x_1 = 1$ and $x_2 = -1$ to $w_1 = 0$, $w_2 = a$. Express your result in terms of the quantity

$$\gamma = \int_{-1}^{1} d\hat{z} \frac{1}{(1-\hat{z})^{2/3}} = \frac{\sqrt{\pi}}{2^{1/4}} \frac{\Gamma(1/3)}{\Gamma(5/6)} \approx 4.20655.$$

(*iii*) (10 marks) Verify for the function $f(z)$ constructed in (*ii*) that the points $x \to \pm\infty$ are mapped to the third vertex of the triangle.

(*iv*) (1 mark) Draw the corresponding figure in the w-plane.

Turn over ...

3) *(i)* (6 marks) Define the exponential growth of a function $f(x)$ and subsequently define the Laplace transform $\mathcal{L}f(x)$ for the function $f(x)$. State the convolution theorem for the Laplace transform.

(ii) (4 marks) For a piecewise smooth function $u(x)$, with reasonable exponential growth, the Laplace transform of its nth derivative $\mathcal{L}u^{(n)}(x)$ may be expressed as

$$\mathcal{L}u^{(n)}(x) = x^n \mathcal{L}u(x) - \sum_{k=0}^{n-1} x^{n-k-1} u^{(k)}(0).$$

Prove this formula for $n = 1$, $n = 2$ and $n = 3$.

(iii) (2 marks) Compute the Laplace transform for the Dirac delta function $v(x) = \delta(x - \alpha)$ and the exponential function $w(x) = e^{-\alpha x}$, with α being a real number.

(iv) (13 marks) Using Laplace transforms, solve the first order differential equation

$$u'(x) + u(x) = 5H(x) - 3H(x - \pi)$$

with boundary condition $u(0) = 5$. The function $H(x)$ is the Heaviside function.

4) The Fourier transform $\mathcal{F}u(x) = \hat{u}(x)$ of a piecewise smooth and absolutely integrable function $u(x)$ on the real line is defined as

$$\mathcal{F}u(x) := \hat{u}(x) = \int_{-\infty}^{\infty} u(t)e^{-itx}dt.$$

(i) (5 marks) Define the convolution $u \star v(x)$ of two functions $v(x)$ and $u(x)$. Then show that the Fourier transform of the convolution of two functions $v(x)$ and $u(x)$ equals the product of the Fourier transforms of $v(x)$ and $u(x)$.

(ii) (3 marks) Compute the Fourier transforms $\mathcal{F}u(x)$ of the function

$$u(x) = e^{-x^2}.$$

You may use the integral $\int_{-\infty}^{\infty} e^{-(t+ix/2)^2}dt = \sqrt{\pi}$.

(ii) (13 marks) Use the Fourier transformation method to show that the solution for the Schrödinger equation

$$i\frac{\partial\psi(x,t)}{\partial t} + \frac{\partial^2\psi(x,t)}{\partial x^2} = 0$$

with initial condition $\psi(x, 0) = g(x)$ and asymptotic behaviour

$$\lim_{x\to\pm\infty} \psi(x,t) = 0, \; \lim_{x\to\pm\infty} \partial\psi(x,t)/\partial x = 0$$

is given by

$$\psi(x,t) = \frac{1}{2\sqrt{\pi i t}} \int_{-\infty}^{\infty} dy g(y) e^{\frac{i}{4t}(x-y)^2}.$$

Turn over ...

5) Consider two infinite cylinders placed non-coaxially with $|z - x_0| = x_0$ and $|z| = 1$. The cylinders are at constant potential $\phi_1 = 0V$ at $|z| = 1$ and $\phi_0 = 50V$ at $|z - x_0| = x_0$. The value of the center of the smaller cylinder and its radius is taken to be $x_0 = 4/17$. Find the potential in the xy-plane between the two cylinders.

(i) (2 marks) Draw the corresponding figure.

(ii) (4 marks) Verify first that the potential for two infinitely long concentric cylinders with radii r_0 and $r = 1$ at potentials ϕ_0, ϕ_1 is given by

$$\phi(r) = (\phi_0 - \phi_1)\frac{\ln r}{\ln r_0} + \phi_1.$$

Hint: The Laplace equation in polar coordinates is

$$\Delta\phi = \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\vartheta^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} = 0.$$

(iii) (16 marks) Verify that the map

$$w = f(z) = \frac{z - \kappa}{\kappa z - 1} \qquad \text{with } \kappa \in \mathbb{R}$$

leaves the circle with the radius $|z| = 1$ invariant, that is the image is $|w| = 1$. Find the constants the constant κ and r_0 such that the circle $|z - x_0| = x_0$ is mapped onto $|w| = r_0$.

(iv) (3 marks) Use the results from (ii) and (iii) to compute the potential for the non-coaxial cylinders.

Internal Examiner:	Professor A. Fring
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Mathematical Methods II (2009)

Solutions and marking scheme:

1) i) • The cross ratio (z_1, z_2, z_3, z_4) is the image of z_4 which maps the points (z_1, z_2, z_3) onto $(0, 1, \infty)$

$$T_c(z_4) = \frac{(z_4 - z_1)(z_2 - z_3)}{(z_4 - z_3)(z_2 - z_1)}$$

• The cross ratio (z_1, z_2, z_3, z_4) is real if and only if the four points z_1, z_2, z_3, z_4 lie on a circle or a line.

• $$T_c(z_4) = \frac{(z_4 - 3)(-3 - 3i)}{(z_4 - 3i)(-3 - 3)} = \frac{(z_4 - 3)(1+i)}{2(z_4 - 3i)}$$

Compute $T_c(1 + i\sqrt{8}) = \dfrac{(1+i\sqrt{8} - 3)(1+i)}{2(1+i\sqrt{8}) - 6i} = -1 - \sqrt{2} \in \mathbb{R} \Rightarrow z_1$ is on the circle $|z| = 3$

$T_c(3+i) = \dfrac{i(1+i)}{2(3-2i)} = \dfrac{1}{26}(-5+i) \notin \mathbb{R} \Rightarrow z_2$ is not on the circle $|z| = 3$. 5

ii). translation: $a = 1,\ b = \Delta,\ c = 0,\ d = 1 \qquad T(z) \to f_T^{\Delta}(z) = z + \Delta$

. rotation : $a = z_0,\ b = 0,\ c = 0,\ d = 1 \qquad T(z) \to f_R^{z_0}(z) = z_0 z$

(this is a rotation + a simultaneous scaling, but usually referred to as rotation)

• inversion map: $a = 0,\ b = 1,\ c = 1,\ d = 0 \qquad T(z) \to f_I(z) = \dfrac{1}{z}$

• for $ad - bc = 0$ $T(z)$ becomes a constant as can be seen from

$$T'(z) = \frac{a}{cz+d} - \frac{(az+b)c}{(cz+d)^2} = \frac{a(cz+d) - c(az+b)}{(cz+d)^2} = \frac{ad - bc}{(cz+d)^2} = 0$$

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• The LFT maps three distinct points z_1, z_2, z_3 uniquely into three distinct points w_1, w_2, w_3. The map is determined by the equation

$$\frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}$$

Substituting $z_1, z_2, z_3, w_1, w_2, w_3$ into this equation

$$\frac{(w+1)}{(w+i)} = \frac{(z-i)(-1)}{(z-1)(-i)}$$

$$\Rightarrow \quad (w+1) = \frac{(z-i)}{i(z-1)}(w+i) \quad \Rightarrow w\left(1+i\,\frac{z-i}{z-1}\right) = \frac{z-i}{z-1} - 1$$

$$\Rightarrow \quad w\left(\frac{z-1+iz+1}{z-1}\right) = \frac{z-i-z+1}{z-1} \quad \Rightarrow w\left(\frac{z(1+i)}{z-1}\right) = \frac{1-i}{z-1}$$

$$\Rightarrow \quad \underline{\underline{w}} = \frac{1-i}{1+i}\,\frac{1}{z} = \underline{\underline{-\frac{i}{z}}}$$

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- This map is unique by the theorem quoted.

iii) Circles and lines in the z-plane:

$$\alpha(x^2+y^2) + \beta x + \gamma y + \delta = 0 \qquad (*)$$

with $z = x+iy$, $w = u+iv$;

$$f_I = w = \frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{(x+iy)(x-iy)} = \underbrace{\frac{x}{x^2+y^2}}_{u} + i\,\underbrace{\frac{-y}{x^2+y^2}}_{v}$$

$$z = \frac{1}{w} = \underbrace{\frac{u}{u^2+v^2}}_{x} + i\,\underbrace{\frac{-v}{u^2+v^2}}_{y}$$

The equation $(*)$ is mapped into

$$\alpha\left(\frac{u^2}{(u^2+v^2)^2} + \frac{v^2}{(u^2+v^2)^2}\right) + \beta\frac{u}{u^2+v^2} - \gamma\frac{v}{u^2+v^2} + \delta = 0 \quad \Big|\cdot(u^2+v$$

$$\Rightarrow \alpha + \beta u - \gamma v + \delta(u^2+v^2) = 0$$

Circles are mapped into circles for $\alpha \neq 0,\ \delta \neq 0$ 9

Circles are mapped into lines for $\alpha \neq 0,\ \delta = 0$ $\boxed{\Sigma = 25}$

Lines are mapped into lines for $\alpha = 0,\ \delta = 0$

Lines are mapped into circles for $\alpha = 0,\ \delta \neq 0$

2) i) Given an n-sided polygon with vertices w_i and exterior angles $\theta_i = \mu_i \pi$ for $1 \le i \le n$, Then there exist always n real numbers x_i for $1 \le i \le n$ together with a complex number $c \in \mathbb{C}$ and an analytic function $f: z \to w$ whose derivative is given by

$$f'(z) = c \prod_{i=1}^{n-1} (z - x_i)^{-\mu_i} \qquad -1 < \mu_i < 1 \quad c \in \mathbb{C}$$

which maps the upper half plane one-to-one onto the interior of the polygon. The points are mapped as $w_i = f(x_i)$ for $1 \le i \le n-1$ and $w_n = \lim_{x \to \pm\infty} f(x)$.

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ii) The exterior angles for an equilateral triangle are $\theta_i = \frac{2\pi}{3}$ $i = 1, 2, 3$.

$$\Rightarrow \qquad f'(z) = c \prod_{i=1}^{2} (z - x_i)^{-\frac{2}{3}}$$

with $x_1 = 1$ and $x_2 = -1$

$$\Rightarrow \qquad f'(z) = c\,(z+1)^{-\frac{2}{3}} (z-1)^{-\frac{2}{3}}$$

integrating

$$\Rightarrow \qquad f(z) = c \int_1^z d\tilde{z}\, (\tilde{z}+1)^{-\frac{2}{3}} (\tilde{z}-1)^{-\frac{2}{3}} + k \qquad k \in \mathbb{C}.$$

$$\because\ f(1) = w_1 = 0 \quad \Rightarrow \quad \underline{k = 0}$$

$$f(-1) = c \int_1^{-1} d\tilde{z}\, (\tilde{z}+1)^{-\frac{2}{3}} (\tilde{z}-1)^{-\frac{2}{3}} = c \int_1^{-1} d\tilde{z}\, (1-\tilde{z}^2)^{-\frac{2}{3}} e^{-i\pi\frac{2}{3}}$$

$$= c\, e^{-\frac{i\pi 2}{3} + i\pi} \int_{-1}^{1} d\tilde{z}\, \frac{1}{(1-\tilde{z}^2)^{\frac{2}{3}}} = c\, e^{\frac{i\pi}{3}} \gamma = w_2 = a$$

$$\Rightarrow \quad \underline{c = \frac{a\, e^{-\frac{i\pi}{3}}}{\gamma}} \qquad \Rightarrow \quad \underline{\underline{f(z) = \frac{a\, e^{-\frac{i\pi}{3}}}{\gamma} \int_1^z d\tilde{z}\, (\tilde{z}^2 - 1)^{-\frac{2}{3}}}}$$

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We should have

$$w_3 = \lim_{z \to -\infty} \frac{a e^{-\frac{i\pi}{3}}}{\gamma} \int_1^z d\bar{z} \left(\bar{z}^2 - 1\right)^{-\frac{2}{3}} \qquad (1)$$

$$w_3 = \lim_{z \to +\infty} \frac{a e^{-\frac{i\pi}{3}}}{\gamma} \int_1^z d\bar{z} \left(\bar{z}^2 - 1\right)^{-\frac{2}{3}} \qquad (2)$$

$$(1): \; w_3 = \frac{a e^{-\frac{i\pi}{3}}}{\gamma} \int_1^{-1} d\bar{z} \left(\bar{z}^2 - 1\right)^{-\frac{2}{3}} + \frac{a e^{-\frac{i\pi}{3}}}{\gamma} \int_{-1}^{-\infty} d\bar{z} \left(\bar{z}^2 - 1\right)^{-\frac{2}{3}}$$

$$= w_2 + \frac{a e^{-\frac{i\pi}{3}}}{\gamma} \int_{-1}^{-\infty} d\bar{z} \, |\bar{z}-1|^{-\frac{2}{3}} \, |\bar{z}+1|^{-\frac{2}{3}} \, e^{-i\pi\frac{4}{3}}$$

$$= w_2 + \frac{a e^{-\frac{i\pi 5}{3}}}{\gamma} (-1) \int_1^{\infty} d\bar{z} \, |\bar{z}-1|^{-\frac{2}{3}} \, |\bar{z}+1|^{-\frac{2}{3}}$$

$$= a + a e^{-\frac{i\pi}{3}} \underbrace{\lim_{z \to \infty} \frac{a e^{-\frac{i\pi}{3}}}{\gamma} \int_1^z d\bar{z} \, |\bar{z}^2 - 1|^{-\frac{2}{3}}}_{w_3 \text{ by } (2)}$$

$$\Rightarrow \underline{\underline{w_3}} = \frac{a}{1 - e^{-\frac{i\pi}{3}}} = \frac{a e^{-\frac{i\pi}{3}}}{e^{-\frac{i\pi}{3}} - e^{-\frac{2i\pi}{3}}} = \frac{a e^{-\frac{i\pi}{3}}}{e^{-\frac{i\pi}{3}} + e^{+\frac{i\pi}{3}}} = \frac{a e^{-\frac{i\pi}{3}}}{2\cos\frac{\pi}{3}} = \underline{\underline{a e^{-\frac{i\pi}{3}}}}$$

Thus w_3 is a vertex of the triangle.

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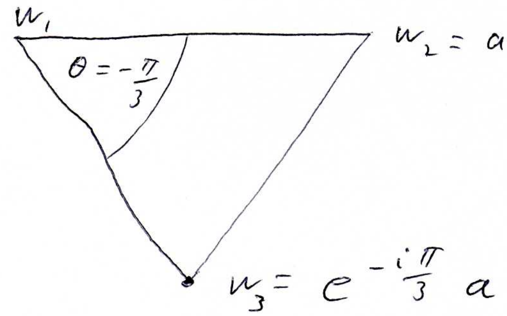
iv)



1

$\boxed{\Sigma = 25}$

3) i) • A function $f(x)$ is said to have exponential growth α if there exists a constant μ such that

$$|u(x)| \leq \mu e^{\alpha x} \qquad \text{for } x > 0 \qquad \mu, \alpha \in \mathbb{R}$$

• The Laplace transform $\mathcal{L}u(x)$ of a piecewise smooth function $u(x)$ with exponential growth α is defined as

$$\mathcal{L}u(x) = \int_0^\infty u(t) e^{-tx} dx \qquad \text{for } x > \alpha$$

- The Laplace transform of the convolution of two functions u and v, i.e. $u * v(x) = \int_{-\infty}^{+\infty} u(t)\, v(x-t)\, dt$ equals the product of the Laplace transforms of these functions

$$\mathcal{L}(u * v)(x) = \mathcal{L}u(x)\, \mathcal{L}v(x).$$ 6

ii) $$\mathcal{L}u'(x) = \int_0^{\infty} u'(t)\, e^{-tx}\, dt = u(t)\, e^{-tx}\Big|_0^{\infty} + x \int_0^{\infty} u(t)\, e^{-tx}\, dt$$

$$= x\, \mathcal{L}u(x) - u(0)$$

$$\Rightarrow \mathcal{L}u''(x) = x\, \mathcal{L}u'(x) - u'(0) = x^2\, \mathcal{L}u(x) - x\, u(0) - u'(0)$$

$$\Rightarrow \mathcal{L}u'''(x) = x\, \mathcal{L}u''(x) - u''(0) = x^3\, \mathcal{L}u(x) - x^2 u(0) - x\, u'(0) - u''(0)$$ 4

iv)

$$u'(x) + u(x) = 5\, H(x) - 3\, H(x-\pi) \qquad u(0) = 5$$

$$\Rightarrow \mathcal{L}u'(x) + \mathcal{L}u(x) = 5\, \mathcal{L}H(x) - 3\, \mathcal{L}H(x-\pi) \qquad \mathcal{L}H(x-d) = \frac{1}{x} e^{-d\cdot}$$

$$\Rightarrow x\, \mathcal{L}u(x) - u(0) + \mathcal{L}u(x) = 5 \frac{1}{x} - 3 \frac{1}{x} e^{-\pi x}$$

from ii)

$$\Leftrightarrow \mathcal{L}u(x)(x+1) - 5 = \frac{1}{x}\left(5 - 3e^{-\pi x}\right)$$

$$\Rightarrow \mathcal{L}u(x) = \frac{1}{1+x}\left(\frac{1}{x}\left(5 - 3e^{-\pi x}\right) + 5\right]$$

$$= \frac{5}{x(1+x)} - \frac{3e^{-\pi x}}{x(1+x)} + \frac{5}{1+x}$$

$$= \frac{5}{x} - 3\frac{e^{-\pi x}}{x} + \frac{3e^{-\pi x}}{1+x}$$

$$\Rightarrow \mathcal{L}u(x) = 5\, \mathcal{L}H(x) - 3\, \mathcal{L}v(x)\, \mathcal{L}H(x) + 3\, \mathcal{L}v(x)\, \mathcal{L}w(x)$$

$\Rightarrow$ for $v(x) = \delta(x-\pi)$ and $w(x) = e^{-x}$

$$\Rightarrow \quad \mathcal{L}u(x) = 5\mathcal{L}H(x) - 3\mathcal{L}v*H(x) + 3\mathcal{L}v*w(x)$$

$$\Rightarrow \quad u(x) = 5H(x) - 3v*H(x) + 3v*w(x)$$

$$\Rightarrow \quad u(x) = 5H(x) - 3\int_{-\infty}^{+\infty} \delta(t-\pi)H(x-t)\,dt + 3\int_{-\infty}^{+\infty}\delta(t-\pi)e^{-(x-t)}\,dt$$

$$u(x) = 5H(x) - 3H(x-\pi) + 3e^{-(x-\pi)}$$

$$u(x) = \begin{cases} 5 + 3e^{-(x-\pi)} & \text{for } 0 < x < \pi \\ 2 + 3e^{-(x-\pi)} & \text{for } \pi \leq x < \infty \end{cases}$$

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iii)

$$\mathcal{L}v(x) = \int_0^\infty \delta(t-\alpha)e^{-tx}\,dt = e^{-\alpha x} \quad \text{for } \alpha > 0$$

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$$\mathcal{L}w(x) = \int_0^\infty e^{-\alpha t}e^{-tx}\,dt = \int_0^\infty e^{(\alpha-x)t}\,dt = \left.\frac{e^{(\alpha-x)t}}{\alpha-x}\right|_0^\infty = \frac{1}{x-\alpha}$$

$\boxed{\Sigma = 25}$ for $x > \alpha$

4) i)

$$u(x) = e^{-x^2}(1+4x)(5-4x^2) = e^{-x^2}(-16x^3-4x^2+20x+5)$$

Express $u(x)$ in terms of $v(x) = e^{-x^2}$ and its derivatives

$$v(x) = e^{-x^2}$$

$$v'(x) = -2x\,e^{-x^2}$$

$$v''(x) = -2e^{-x^2} + 4x^2e^{-x^2}$$

$$v'''(x) = 12x e^{-x^2} - 8x^3e^{-x^2}$$

$$\Rightarrow \quad u(x) = \alpha v(x) + \beta v'(x) + \gamma v''(x) + \delta v'''(x)$$

$$= \left(\alpha - 2x\beta - 2\gamma + 4x^2\gamma + 12x\delta - 8x^3\delta\right)e^{-x^2}$$

$$= \left[(\alpha-2\gamma) + (12\delta - 2\beta)x + 4\gamma x^2 - 8\delta x^3\right]e^{-x^2}$$

$$= (5 + 20x - 4x^2 - 16x^3)\,e^{-x^2}$$

$$\Rightarrow \quad \alpha - 2\gamma = 5, \qquad 12\delta - 2\beta = 20, \qquad -4 = 4\gamma, \qquad -16 = -8\delta .$$

$$\Rightarrow \quad \underline{\gamma = -1} \qquad \underline{\delta = 2} \qquad \underline{\alpha = 3} \qquad \beta = 2$$

$$\Rightarrow \quad u(x) = 3v(x) + 2v'(x) - v''(x) + 2v'''(x)$$

$$\Rightarrow \quad \mathcal{F}u(x) = 3\mathcal{F}v(x) + 2\underbrace{\mathcal{F}v'(x)}_{ix\mathcal{F}v(x)} - \underbrace{\mathcal{F}v''(x)}_{ix\underbrace{\mathcal{F}v'(x)}_{ix\mathcal{F}v(x)}} + 2\underbrace{\mathcal{F}v'''(x)}_{ix\underbrace{\mathcal{F}v''(x)}_{ix\underbrace{\mathcal{F}v'(x)}_{ix\mathcal{F}v(x)}}}$$

$$= \left(3 + 2ix + x^2 - 2ix^3\right)\mathcal{F}v(x)$$

$$\Rightarrow \quad \boxed{\mathcal{F}u(x) = \sqrt{\pi}\left(3 + 2ix + x^2 - 2ix^3\right)e^{-\frac{x^2}{4}}}$$

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(ii) Solve $i\psi_t(x,t) + \psi_{xx}(x,t) = 0$ (1) with $\psi(x,0) = \delta(x)$ (2)

$\lim_{x\to\pm\infty} \psi(x,t) = 0$ (3)

Fourier transform in x:

$$\hat{\psi}(x,t) = \mathcal{F}_x\psi(x,t) = \int_{-\infty}^{+\infty} \psi(s,t)e^{-isx}\,ds$$

$\lim_{x\to\pm\infty} \psi_x(x,t) = 0$ (4)

$$\Rightarrow \quad i\,\partial_t\hat{\psi}(x,t) = i\int_{-\infty}^{+\infty} \partial_t\psi(s,t)e^{-isx}\,ds$$

$$= -\int_{-\infty}^{+\infty} \psi_{ss}(s,t)e^{-isx}\,ds \quad (\text{with } (1))$$

$$= \underbrace{-\psi_s(s,t)e^{-isx}\Big|_{-\infty}^{+\infty}}_{0 \text{ by } (4)} - ix\int_{-\infty}^{+\infty} \psi_s(x,t)e^{-isx}\,ds \quad \left(\begin{matrix}\text{int.}\\ \text{by}\\ \text{parts}\end{matrix}\right)$$

$$= \underbrace{-ix\,\psi(s,t)e^{-isx}\Big|_{-\infty}^{+\infty}}_{0 \text{ by } (4)} + x^2\underbrace{\int_{-\infty}^{+\infty} \psi(s,t)e^{-isx}\,ds}_{\hat{\psi}(x,t)}$$

$$\Rightarrow \quad i\,\partial_t\hat{\psi}(x,t) = x^2\hat{\psi}(x,t)$$

$$\Rightarrow \quad \text{solution} \quad \hat{\psi}(x,t) = e^{-itx^2}\hat{\delta}(x) \qquad \text{with } (2)$$

with $\mathcal{F}\, a(x) = \sqrt{\pi}\, e^{-\frac{x^2}{4}}$ $\qquad a(x) = e^{-x^2}$

$\Rightarrow \mathcal{F}\, v(x) = 2\sqrt{\pi t}\, e^{-t x^2}$ $\qquad v(x) = e^{-\frac{x^2}{4t}}$

$\Rightarrow \mathcal{F}\, w(x) = e^{-i t x^2}$ $\qquad w(x) = \frac{1}{2\sqrt{i\pi t}}\, e^{i\frac{x^2}{4t}}$

$$\Rightarrow \hat{\psi}(x,t) = \hat{w}(x,t)\,\hat{\delta}(x)$$

$$\Leftrightarrow \mathcal{F}_x \psi(x,t) = \mathcal{F}_x w(x,t)\, \mathcal{F}_x \delta(x) = \mathcal{F}_x \left(w(x,t) * \delta(x) \right) \quad \Big| \, \mathcal{F}_x^{-1}$$

$$\Rightarrow \underline{\underline{\psi(x,t)}} = w(x,t) * \delta(x) = \underline{\underline{\int_{-\infty}^{+\infty} dy\, \delta(y)\, \frac{1}{2\sqrt{i\pi t}}\, e^{\frac{i}{4t}(x-y)^2}}}$$

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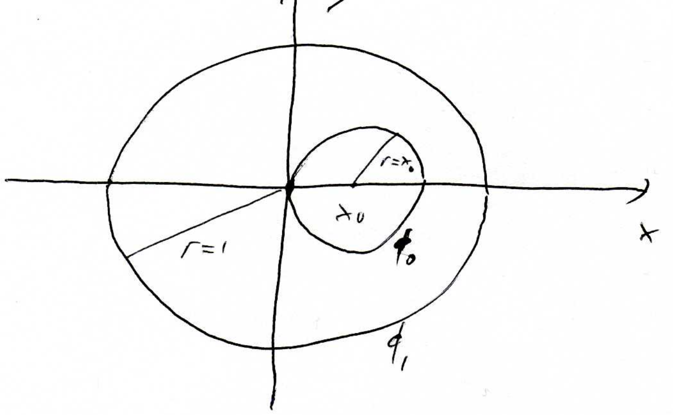
$\boxed{\Sigma = 25}$

5) i)



2

ii) As there is no ϑ dependence we have $\partial\phi/\partial\vartheta = 0$.

$$\Rightarrow \quad \Delta\phi = \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\vartheta^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} = 0$$

becomes

$$r\frac{\partial^2\phi}{\partial r^2} + \frac{\partial\phi}{\partial r} = 0$$

For $\phi(r) = (\phi_0 - \phi_1)\frac{\ln r}{\ln r_0} + \phi_1$

$$\left.\begin{aligned} \frac{\partial\phi(r)}{\partial r} &= (\phi_0 - \phi_1)\frac{1}{\ln r_0}\frac{1}{r} \\ \frac{\partial^2\phi(r)}{\partial r^2} &= -(\phi_0 - \phi_1)\frac{1}{\ln r_0}\frac{1}{r^2} \end{aligned}\right\} \Rightarrow \Delta\phi = 0$$

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iii) The map is a linear fractional transformation,

i.e. three points are mapped uniquely into three points.

Take three points $(z_1 = -1,\ z_2 = -1,\ z_3 = i) \in |z| = 1$

Compute $\quad f(1) = -1 \in |w| = 1$

$$f(-1) = 1 \in |w| = 1$$

$$f(i) = \frac{i - \kappa}{i\kappa - 1} = \frac{(i-\kappa)(-i\kappa-1)}{(i\kappa-1)(-i\kappa-1)} = \frac{2\kappa}{1+\kappa^2} + i\,\frac{(\kappa^2-1)}{1+\kappa^2}$$

We verify $\left(\operatorname{Re} f(i)\right)^2 + \left(\operatorname{Im} f(i)\right)^2 = 1 \quad \Rightarrow \quad f(i) \in |w| = 1$

We have (inspect figure)

$$f(0) = r_0 \qquad \Rightarrow \qquad \kappa = r_0$$

$$f(2x_0) = -r_0 \qquad \Rightarrow \qquad \frac{2x_0 - r_0}{2x_0 r_0 - 1} = -r_0$$

Therefore

$$2x_0 - r_0 + 2x_0 r_0^2 - r_0 = 0 \quad \Rightarrow \quad \underline{r_{0\,\frac{1}{2}} = \frac{1}{2x_0} \pm \frac{1}{2x_0}\sqrt{1-4x_0^2}}$$

for $x_0 = \frac{4}{17}$

$$(r_0)_{\frac{1}{2}} = \frac{17}{2\cdot 4} \pm \frac{17}{2\cdot 4}\sqrt{1 - 4\left(\frac{4}{17}\right)^2} \quad \Rightarrow \quad \underline{(r_0)_1 = 4}, \quad \underline{(r_0)_2 = \frac{1}{4}}$$

$\because \ r_0 < 1 \quad \Rightarrow \quad$ discard $(r_0)_1$

$$\Rightarrow \quad \underline{f(z) = \frac{z - \frac{1}{4}}{\frac{1}{4}z - 1} = \frac{4z-1}{z-4}}$$

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iv) $\quad \underline{\phi(r)} = \phi_{COAXIAL}\left(f(r)\right) = (\phi_0 - \phi_1)\,\frac{\ln\left(f(r)\right)}{\ln r_0} = \underline{-50\,\frac{\ln\left|\frac{4z-1}{z-4}\right|}{\ln 4}\ V}$

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$\boxed{\Sigma = 25}$