MA 3605

CITY UNIVERSITY

London

BSc Degrees in Mathematical Science
Mathematical Science with Statistics
Mathematical Science with Computer Science
Mathematical Science with Finance and Economics
MMath Degrees in Mathematical Science

Part III Examination

Mathematical Methods

May 2010

Time allowed: 2 hours

Full marks may be obtained for correct answers to THREE of the FIVE questions.

If more than THREE questions are answered, the best THREE marks will be credited.

Turn over ...

Each question carries 25 marks.

1) (*i*) (2 marks) The linear fractional transformation is defined as

$$w = T(z) = \frac{az + b}{cz + d} \qquad \text{for } ad - bc \neq 0; a, b, c, d \in \mathbb{C}.$$

For which choices of the constants a, b, c, d does this map reduce to a translation by an amount Δ, a rotation of 90 degrees in the positive mathematical sense, a scaling by a factor 2 and the inversion map?

(*ii*) (5 marks) Given the two linear fractional transformations

$$T_1(z) = \frac{z - 3i}{z + i} \qquad \text{and} \qquad T_2(z) = \frac{z + 2}{2z - 3}$$

compute the linear fractional transformation equivalent to the composition

$$T_2 \circ T_1(z).$$

Why is your answer expected to be a linear fractional transformation? Decompose your result into a succession of rotations, translations and inversions.

(*iii*) (8 marks) Determine the linear fractional transformation $T(z)$, which maps the points $z_1 = 2i, z_2 = 1 + i, z_3 = 0$ in the z-plane onto $w_1 \to \infty, w_2 = 1, w_3 = i$ in the w-plane. Present your answer in the form as specified in (*i*). Is this map unique? Subsequently find the map which maps $z_1 \to \infty, z_2 = 1, z_3 = i$ into $w_1 = 2i, w_2 = 1 + i, w_3 = 0$.

(*iv*) (10 marks) Verify that for a certain range of values for λ the function

$$f(z) = \frac{\lambda}{2}\left(z + \frac{1}{z}\right) \qquad \text{for } \lambda \in \mathbb{C},$$

maps the exterior of a semicircle in the upper half plane with radius one centered at the origin onto the lower half plane. Sketch the corresponding figure. Quote the relevant theorem to argue that in general for a specific choice of λ this map is unique.

Turn over . . .

2) (*i*) (3 marks) State the Schwarz-Christoffel theorem.

(*ii*) (11 marks) Determine the Schwarz-Christoffel transformation $f(z)$, which maps the upper half plane onto an isosceles triangle. Map the points $x_1 = 1$ and $x_2 = -1$ to the two points $w_1 = 0$, $w_2 = a \in \mathbb{R}^+$, where the interior angles of the triangle are both $\pi/6$ at these points.

Hint: You may use the integral

$$\int_{-1}^{1} dt \frac{1}{(1-t^2)^{5/6}} = \sqrt{\pi}\frac{\Gamma(1/6)}{\Gamma(2/3)}.$$

(*iii*) (10 marks) Determine the position of the third vertex of the triangle in the w-plane using the fact that $w_3 = \lim_{z\to\pm\infty} f(z)$.

(*iv*) (1 mark) Draw the corresponding figure in the w-plane.

3) (*i*) (12 marks) Define the convolution $u \star v(x)$ of two functions $v(x)$ and $u(x)$. Then show that the Laplace transform of the convolution of two functions $v(x)$ and $u(x)$ equals the product of the Laplace transforms of $v(x)$ and $u(x)$.

(*ii*) (13 marks) Use the Laplace transformation method to solve the following ordinary differential equation

$$y''(x) + 2y'(x) + 5y(x) = 3e^{-x}\cos x.$$

The boundary conditions are $y(0) = 2$ and $y'(0) = -2$. The dashes indicate derivatives with respect to x.

Hints: You may use the fact that

$$\mathcal{L}y'(x) = x\mathcal{L}y(x) - y(0)$$

and

$$\mathcal{L}(e^{-x}\cos\lambda x) = \frac{1+x}{(x+1)^2+\lambda^2} \qquad \text{for } \lambda \in \mathbb{R}.$$

4) The Fourier transform $\mathcal{F}u(x) = \hat{u}(x)$ of a piecewise smooth and absolutely integrable function $u(x)$ on the real line is defined as

$$\mathcal{F}u(x) := \hat{u}(x) = \int_{-\infty}^{\infty} u(t)e^{-itx}dt.$$

(i) (4 marks) Define precisely what is meant by a *piecewise smooth function* and an *absolutely integrable function.* Provide an example for a function which is not piecewise smooth and also an example for a function which is not absolutely integrable.

(ii) (3 marks) Compute the Fourier transform $\mathcal{F}u(x)$ of the function

$$u(x) = e^{-x^2}.$$

You may use the integral $\int_{-\infty}^{\infty} e^{-(t+ix/2)^2}dt = \sqrt{\pi}$.

(iii) (9 marks) Use the relation between the Fourier transform and the Fourier transform of its derivative $\mathcal{F}u'(x) = ix\mathcal{F}u(x)$ together with the result of *(ii)* to compute the Fourier transform $\mathcal{F}u(x)$ of the function

$$u(x) = 2P_3(x)e^{-x^2},$$

where $P_3(x)$ is a Legendre polynomial of order three. The Legendre polynomial of order n is generated from

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left(x^2 - 1\right)^n.$$

(iv) (9 marks) Use the Fourier transformation method to solve the following integral equation for $\phi(x)$

$$h(x) = \int_{-\infty}^{\infty} g(x-s)\phi(s)ds,$$

where

$$h(x) = \begin{cases} 1 & \text{for } |x| \leq 2 \\ 0 & \text{for } |x| > 2 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 1 & \text{for } |x| \leq 1 \\ 0 & \text{for } |x| > 1 \end{cases}.$$

Hint: You may use $\mathcal{F}\delta(x+\lambda) = e^{i\lambda x}$, where $\delta(x)$ denotes the Dirac delta function.

Turn over ...

5) (*i*) (22 marks) Use a Laplace transform in t to solve the wave equation

$$\phi_{xx}(x,t) = \phi_{tt}(x,t)$$

subject to the initial conditions $\phi(x,0) = \cos x$ for $0 < x < \pi$, $\phi_t(x,0) = 0$ and $\phi(\pi/2,t) = 0$ for $t > 0$.

Hint: You may use the fact that

$$\mathcal{L}y'(x) = x\mathcal{L}y(x) - y(0)$$

and

$$\mathcal{L}(\cos \lambda x) = \frac{x}{x^2 + \lambda^2}.$$

(*ii*) (3 marks) Verify your solution.

Internal Examiner: Professor A. Fring
External Examiners: Professor J. Billingham
Professor E. Corrigan

MA3605 Mathematical Methods II

Solutions and marking scheme for exam May 2010

INSTRUCTIONS: Full marks are obtained for correct answers to three of the five questions. Each question carries 25 marks.
(All questions and aswers are unseen. Definitions, theorems, the proof in 3(i) and 4(ii) have been seen.)

1. (i) The notations of the lecture are $\boxed{2}$

$$f_T^{\Delta}(z) := z + \Delta, \qquad f_R^{z_0}(z) := zz_0 \qquad \text{and} \qquad f_I(z) := \frac{1}{z},$$

where it is understood that $f_R^{z_0}$ is a simultaneous scaling and rotation. With

$$T(z) = \frac{az+b}{cz+d}$$

we therefore have

$$\begin{array}{ll} T(z) \to f_T^{\Delta}(z) & \text{for } a=1,\, b=\Delta,\, c=0,\, d=1, \\ T(z) \to f_R^{e^{i\pi/2}}(z) & \text{for } a=e^{i\pi/2},\, b=0,\, c=0,\, d=1, \\ T(z) \to f_R^{e^{i\pi/2}}(z) & \text{for } a=2,\, b=0,\, c=0,\, d=1, \\ T(z) \to f_I(z) & \text{for } a=0,\, b=1,\, c=1,\, d=0. \end{array}$$

(ii) With $\boxed{5}$

$$T_1(z) = \frac{z-3i}{z+i} \qquad \text{and} \qquad T_2(z) = \frac{z+2}{2z-3}$$

we compute

$$T_2 \circ T_1(z) = \frac{\frac{z-3i}{z+i}+2}{2\frac{z-3i}{z+i}-3} = \frac{z-3i+2z+2i}{2z-6i-3z-3i} = \frac{3z-i}{-z-9i} = \frac{-3z+i}{z+9i}.$$

The result is expected to a be a also linear fractional transformation as the set of all linear fractional transformations forms a *group*. By the closure property of the group we expect $T_2 \circ T_1$ to be a linear fractional transformation.
We can decompose this as

$$\begin{aligned} T_2 \circ T_1(z) &= \frac{-3(z+9i)+27i+i}{z+9i} = -3 + \frac{28i}{z+9i} \\ &= f_T^{-3} \circ f_R^{28i} \circ f_I \circ f_T^{9i}(z). \end{aligned}$$

(*iii*) We use the following $\boxed{8}$

Theorem : *The linear fractional transformation $w = T(z)$ maps three distinct points z_1, z_2, z_3 uniquely into three distinct points w_1, w_2, w_3. The map is determined by the equation*

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}.$$

First we take the limit

$$\lim_{w_1\to\infty} \frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(w_2-w_3)}{(w-w_3)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}.$$

Substituting the remaining values gives

$$\frac{(1-i)}{(w-i)} = \frac{(z-2i)(1+i-0)}{(z-0)(1+i-2i)}.$$

Solving this for w gives

$$w = T(z) = \frac{2-z}{z-2i}.$$

The remaining map is simply the inverse of $T(z)$. Thus we solve

$$z = \frac{2-w}{w-2i}$$

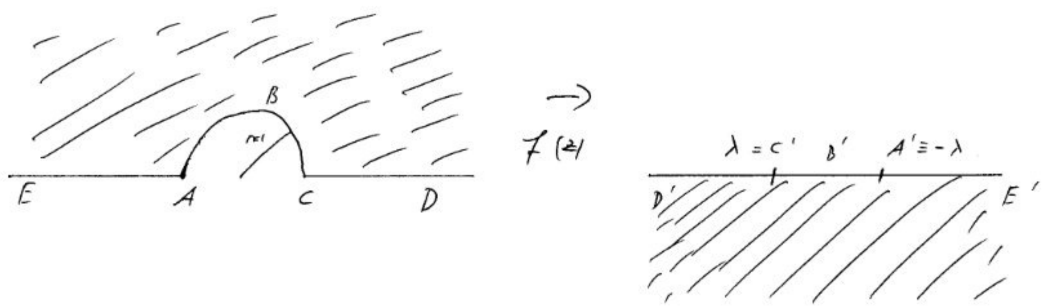
for w, which yields

$$w = T^{-1}(z) = \frac{2iz+1}{z+1}.$$

(*iv*) We have $\boxed{10}$

$$f(z) = \frac{\lambda}{2}\left(z + \frac{1}{z}\right) \qquad \text{for } \lambda \in \mathbb{C}.$$

First we need to verify that the boundaries are mapped correctly:



- The semicircle $\widehat{ABC}$ is parameterized by $z = e^{i\theta}$ for $0 \leq \theta \leq \pi$
  $\Rightarrow f(z) = \lambda/2(e^{i\theta} + e^{-i\theta}) = \lambda \cos\theta$
  $\Rightarrow$ for $\lambda \in \mathbb{R}$ the image of the semicircle is the interval $[-\lambda, \lambda]$
- The interval $(1, \infty) \equiv \overline{CD}$ is parameterized by $z = re^{i\theta}$ for $\theta = 0$ and $r \in (1, \infty)$

$\Rightarrow f(z) = \lambda/2(r + r^{-1}) \in \lambda(1, \infty)$

- The interval $(-\infty, -1) \equiv \overline{EA}$ is parameterized by $z = re^{i\theta}$ for $\theta = \pi$ and $r \in (-\infty, -1)$

$\Rightarrow f(z) = \lambda/2(r + r^{-1}) \in \lambda(-\infty, -1)$

This means the boundaries are mapped correctly.

The Riemann mapping theorem states:

Given a simply connected region $D \subset \mathbb{C}$, which is not the entire plane and a point $z_0 \in D$. Then there exists an analytic function $f : z \mapsto w$ which maps D one-to-one onto the interior of the unit disk $|w| < 1$. The uniqueness of the map can be achieved with the normalization condition $f(z_0) = 0$ and $f'(z_0) > 0$.

As the upper half plane minus the region below the semicircle is simply connected, this means we could have mapped this region either to the lower or the upper half plane. It suffices to verify this for one point. We compute the image for $z = 2i$, i.e. a point in the upper half plane

$$f(2i) = \frac{\lambda}{2}\left(2i + \frac{1}{2i}\right) = \frac{\lambda i}{2}\left(2 - \frac{1}{2}\right) = \frac{3}{4} i\lambda.$$

The image point is only in the lower half plane for $\lambda \in \mathbb{R}^-$.

By the Riemann mapping theorem this proves that the upper half plane minus the region below the semicircle is *uniquely* mapped into the lower half plane. $\boxed{\sum = 25}$

2. (i) Given an n-sided polygon with vertices w_i and exterior angles $\theta_i = \mu_i \pi$ for $1 \leq i \leq n$. Then there exist always n real numbers x_i for $1 \leq i \leq n$ together with a complex constant $c \in \mathbb{C}$ and an analytic function $f : z \mapsto w$ whose derivative is given by $\boxed{3}$

$$f'(z) = c \prod_{i=1}^{n-1} (z - x_i)^{-\mu_i} \qquad c \in \mathbb{C}, -1 < \mu_i < 1, \tag{1}$$

which maps the upper half plane one-to-one onto the interior of the polygon. The points are mapped as $w_i = f(x_i)$ for $1 \leq i \leq n-1$ and $w_n = \lim_{x \to \pm\infty} f(x)$.

(ii) The two exterior angles θ_i with $i = 1, 2$ at the points w_1, w_2 are $\theta_i = 5\pi/6$. We can therefore express the derivative (1) as $\boxed{11}$

$$f'(z) = c \prod_{i=1}^{2} (z - x_i)^{-5/6}. \tag{2}$$

Let us take next the points x_i to be $x_1 = 1$ and $x_2 = -1$, such that

$$f'(z) = c(z-1)^{-5/6}(z+1)^{-5/6}.$$

Integration (2) then yields

$$f(z) = c \int_1^z d\hat{z} (\hat{z} - 1)^{-5/6} (\hat{z} + 1)^{-5/6} + \tilde{c},$$

with $\tilde{c} \in \mathbb{C}$ some integration constant. Let us now fix the constants c and $\tilde{c}$ by substituting the values for all vertices. We have taken here the lower limit to be 1 as this simply yields

$$f(1) = \tilde{c} = w_1 = 0.$$

Furthermore we have

$$f(-1) = c \int_1^{-1} d\hat{z}(\hat{z}-1)^{-5/6}(\hat{z}+1)^{-5/6} = w_2 = a, \qquad (3)$$

$$\lim_{z\to-\infty} f(z) = c \int_1^{-\infty} d\hat{z}(\hat{z}-1)^{-5/6}(\hat{z}+1)^{-5/6} = w_3, \qquad (4)$$

$$\lim_{z\to\infty} f(z) = c \int_1^{\infty} d\hat{z}(\hat{z}-1)^{-5/6}(\hat{z}+1)^{-5/6} = w_3.$$

The constant c is determined from (3)

$$w_2 = ce^{-i\pi 5/6} \int_1^{-1} d\hat{z}(1-\hat{z}^2)^{-5/6} = e^{i\pi/6} c\sqrt{\pi}\frac{\Gamma(1/6)}{\Gamma(2/3)} \quad \Rightarrow \quad c = e^{-i\pi/6}\frac{a}{\sqrt{\pi}}\frac{\Gamma(2/3)}{\Gamma(1/6)}$$

such that

$$f(z) = e^{-i\pi/6}\frac{a}{\sqrt{\pi}}\frac{\Gamma(2/3)}{\Gamma(1/6)} \int_1^z d\hat{z}(z-1)^{-5/6}(z+1)^{-5/6}.$$

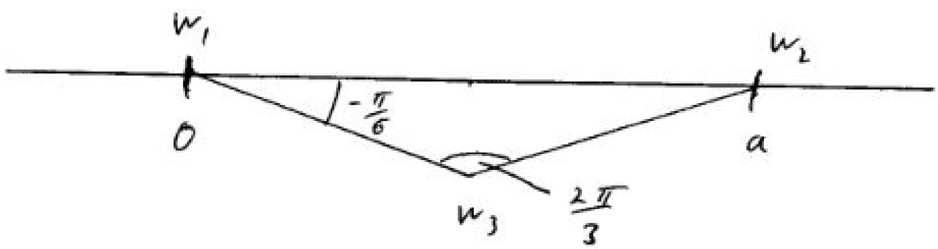
(*iii*) Next we compute (4) $\boxed{10}$

$$\begin{aligned} w_3 &= c\int_1^{-1} d\hat{z}(\hat{z}-1)^{-5/6}(\hat{z}+1)^{-5/6} + c\int_{-1}^{-\infty} d\hat{z}(\hat{z}-1)^{-5/6}(\hat{z}+1)^{-5/6} \\ &= w_2 + ce^{-i\pi 5/3}\int_{-1}^{-\infty} d\hat{z}\,|\hat{z}-1|^{-5/6}\,|\hat{z}+1|^{-5/6} \\ &= w_2 + ce^{-i\pi 2/3}\int_1^{\infty} d\hat{z}\,|\hat{z}-1|^{-5/6}\,|\hat{z}+1|^{-5/6} \\ &= w_2 + e^{-i\pi 2/3} w_3 = a + e^{-i\pi 2/3} w_3. \end{aligned}$$

Solving this for w_3 gives

$$w_3 = \frac{a}{1-e^{-i\pi 2/3}} = \frac{ae^{i\pi/3}}{e^{i\frac{\pi}{3}} - e^{-i\frac{\pi}{3}}} = \frac{ae^{i\pi/3}}{2i\sin\pi/3} = \frac{ae^{i\pi/3}}{i\sqrt{3}} = \frac{ae^{-i\pi/6}}{\sqrt{3}}.$$

(*iv*) w_3 is in the lower half plane $\boxed{1}$



$\boxed{\sum = 25}$

3. (i) The *convolution* of two functions $u(x)$ and $v(x)$ is defined as $\boxed{12}$

$$u \star v(x) = \int_{-\infty}^{\infty} u(t)v(x-t)dt.$$

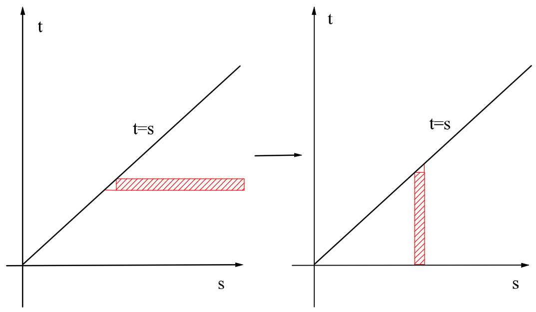
By definition of the Laplace transform we have

$$\begin{aligned}
&(\mathcal{L}u)(x)(\mathcal{L}v)(x) \\
&= \int_0^\infty u(t)e^{-tx}dt \int_0^\infty v(s)e^{-sx}ds \\
&= \int_0^\infty dt \int_0^\infty ds\ u(t)v(s)e^{-x(t+s)}.
\end{aligned}$$

Shifting now $s \to s+t$ we obtain

$$(\mathcal{L}u)(x)(\mathcal{L}v)(x) = \int_0^\infty dt \int_t^\infty ds\ u(t)v(s-t)e^{-xs}.$$

Next we change the order of integration according to $\int_0^\infty dt \int_t^\infty ds \to \int_0^\infty ds \int_0^s dt$. The following figure provides an illustration of this.



To cover the entire integration area we can either sum up horizontal or vertical slices. We then obtain

$$\begin{aligned}
(\mathcal{L}u)(x)(\mathcal{L}v)(x) &= \int_0^\infty ds \int_0^s dt\ u(t)v(s-t)e^{-xs} \\
&= \int_0^\infty ds \left(\int_{-\infty}^\infty dt\ u(t)v(s-t) \right) e^{-xs} \qquad (5) \\
&= \int_0^\infty ds (u \star v)(s) e^{-xs} \\
&= \mathcal{L}(u \star v)(x).
\end{aligned}$$

In (5) we have extended the integration limits to $\pm\infty$, by using the fact that $u(t) = v(t) = 0$ for $t < 0$.

(ii) Acting on the differential equation with $\mathcal{L}$ $\boxed{13}$

$$\mathcal{L}y''(x) + 2\mathcal{L}y'(x) + 5\mathcal{L}y(x) = 3\mathcal{L}(e^{-x}\cos x).$$

Using now the first hint gives

$$\begin{aligned} \mathcal{L}y'(x) &= x\mathcal{L}y(x) - y(0), \\ \mathcal{L}y''(x) &= x\mathcal{L}y'(x) - y'(0) = x^2\mathcal{L}y(x) - xy(0) - y'(0). \end{aligned}$$

The differential equation then becomes

$$x^2\mathcal{L}y(x) - xy(0) - y'(0) + 2[x\mathcal{L}y(x) - y(0)] + 5\mathcal{L}y(x) = 3\mathcal{L}(e^{-x}\cos x).$$

Solving this for $\mathcal{L}y(x)$ and using the second hint

$$\mathcal{L}(e^{-x}\cos x) = \frac{1+x}{(x+1)^2+1} \qquad \text{for } \lambda \in \mathbb{R},$$

yields

$$\begin{aligned} \mathcal{L}y(x) &= 2\frac{1+x}{x^2+2x+5} + 3\frac{1+x}{(x^2+2x+5)[(x+1)^2+1]} \\ &= 2\frac{1+x}{x^2+2x+5} + 3\left[\frac{(1+x)/3}{[(x+1)^2+1]} - \frac{(1+x)/3}{[(x+1)^2+4]}\right] \\ &= \frac{(1+x)}{(x+1)^2+4} + \frac{(1+x)}{(x+1)^2+1} \\ &= \mathcal{L}(e^{-x}\cos 2x) + \mathcal{L}(e^{-x}\cos x) \end{aligned}$$

Acting on this equation with $\mathcal{L}^{-1}$ gives $\boxed{\sum = 25}$

$$y(x) = e^{-x}(\cos x + \cos 2x)$$

4. (*i*) **Definition:** *A function u(x) is said to be* <u>*absolutely integrable*</u> *when* $\boxed{4}$

$$\int_{-\infty}^{\infty} |u(t)|\, dt < \infty.$$

Definition: *A function u(x) is said to be* <u>*piecewise smooth*</u> *when there exist a finite number of points* $x_1 < x_2 < \ldots < x_n$ *on the real axis such that*

a) u(x) is continuous on all the intervals $(-\infty, x_1)$, (x_1, x_2), $\ldots$ (x_n, ∞).

b) the left and right limits of u(x) exists on all points $x_1, x_2, \ldots, x_n$.

For instance, the function

$$u(x) = \begin{cases} \frac{1}{x} & \text{for } x < 0 \\ 0 & \text{for } x > 0 \end{cases}$$

is not piecewise smooth as the left limit at $x = 0$ is not finite.
The *Heavyside function* (unit step function)

$$u(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 & \text{for } x \geq 0 \end{cases}$$

is not absolutely integrable since $\int_{-\infty}^{\infty} |u(t)|\, dt = \int_0^{\infty} dt \to \infty$.

(ii) Clearly $u(x)$ is piecewise smooth. From the definition of the Fourier transform follows $\boxed{3}$

$$\begin{aligned}\mathcal{F}u(x) &= \int\limits_{-\infty}^{\infty} e^{-t^2} e^{-itx} dt = \int\limits_{-\infty}^{\infty} e^{-(t+ix/2)^2} e^{-x^2/4} dt \\ &= e^{-x^2/4} \int\limits_{-\infty}^{\infty} e^{-(t+ix/2)^2} dt = \sqrt{\pi} e^{-x^2/4},\end{aligned}$$

where we used the integral $\int_{-\infty}^{\infty} e^{-(t+ix/2)^2} dt = \sqrt{\pi}$.

(iii) First we compute $\boxed{9}$

$$\begin{aligned}P_3(x) &= \frac{1}{2^3 3!} \frac{d^3}{dx^3} \left(x^2-1\right)^3 \\ &= \frac{1}{48} \frac{d^2}{dx^2} \left[3\left(x^2-1\right)^2 2x\right] \\ &= \frac{1}{8} \frac{d}{dx} \left[2\left(x^2-1\right) 2x^2 + \left(x^2-1\right)^2\right] \\ &= \frac{1}{8} \left[4\left(2xx^2 + \left(x^2-1\right) 2x\right) + 2\left(x^2-1\right) 2x\right] \\ &= \frac{1}{2}\left(5x^3 - 3x\right),\end{aligned}$$

such that

$$u(x) = 2P_3(x) e^{-x^2} = \left(5x^3 - 3x\right) e^{-x^2}.$$

Next we define $v(x) = e^{-x^2}$. Then $v'(x) = -2xe^{-x^2}$, $v''(x) = (4x^2-2)e^{-x^2}$ and $v'''(x) = (12x - 8x^3)e^{-x^2}$. Therefore

$$\begin{aligned}u(x) &= \alpha v(x) + \beta v'(x) + \gamma v''(x) + \delta v'''(x) \\ &= \left[(\alpha - 2\gamma) + (12\delta - 2\beta)x + 4\gamma x^2 - 8\delta x^3)\right] e^{-x^2} \\ &= \left(5x^3 - 3x\right) e^{-x^2}\end{aligned}$$

Comparing coefficients gives

$$\alpha - 2\gamma = 0, \quad 12\delta - 2\beta = -3, \quad 4\gamma = 0 \quad \text{and} \quad -8\delta = 5,$$

which is solved by $\alpha = 0$, $\beta = -9/4$, $\gamma = 0$ and $\delta = -5/8$.Therefore

$$\begin{aligned}\mathcal{F}u(x) &= -\frac{1}{8}\left(9\mathcal{F}v'(x) + 5\mathcal{F}v'''(x)\right) \\ &= -\frac{1}{8}\left(18ix - 5ix^3\right) \mathcal{F}v(x) \\ &= \sqrt{\pi}\frac{i}{8}\left(5x^3 - 18x\right) e^{-x^2/4}.\end{aligned}$$

(iv) First we notice that the right hand side of $\boxed{9}$

$$h(x) = \int_{-\infty}^{\infty} g(x-s)\phi(s)ds,$$

is a convolution between $g(x)$ and $\phi(x)$, i.e. we can rewrite it as

$$h(x) = g \star \phi(x).$$

Acting on this with $\mathcal{F}$ gives

$$\mathcal{F}h(x) = \mathcal{F}\left(g \star \phi\right)(x) = \mathcal{F}g(x)\mathcal{F}\phi(x). \qquad (*)$$

In the last equality we used the convolution theorem. We need to compute $\mathcal{F}h(x)$ and $\mathcal{F}g(x)$:
Consider the function

$$u_\lambda(x) = \begin{cases} 1 & \text{for } |x| < \lambda \\ 0 & \text{for } |x| > \lambda \end{cases},$$

such that $u_1(x) = g(x)$ and $u_2(x) = h(x)$.
This part of the answer is not essential to obtain full marks:
• First we verify that $u(x)$ is piecewise smooth. Except at $x = \pm\lambda$ the function is continuous, such that only at these two point we might encounter a problem. The left and right limits at these points exist. The left limits are

$$\lim_{\varepsilon\to 0} u(\lambda - \varepsilon) = 1 \qquad \lim_{\varepsilon\to 0} u(-\lambda - \varepsilon) = 0$$

and the right limits are

$$\lim_{\varepsilon\to 0} u(\lambda + \varepsilon) = 0 \qquad \lim_{\varepsilon\to 0} u(-\lambda + \varepsilon) = 1.$$

The function $u(x)$ is also absolutely integrable

$$\int_{-\infty}^{\infty} |u(t)|\, dt = \int_{-\lambda}^{\lambda} 1dt = 2\lambda < \infty. \qquad \bullet$$

From the definition of the Fourier transform follows

$$\mathcal{F}u(x) = \int_{-\lambda}^{\lambda} e^{-itx}dt = \frac{i}{x}e^{-itx}\Big|_{-\lambda}^{\lambda} = 2\frac{\sin\lambda x}{x}.$$

This means equation $(*)$ can be written as

$$2\frac{\sin 2x}{x} = 2\frac{\sin x}{x}\mathcal{F}\phi(x)$$

and therefore

$$\mathcal{F}\phi(x) = 2\cos x.$$

Using the hint

$$\mathcal{F}\left[\delta(x+1) + \delta(x-1)\right] = e^{ix} + e^{-ix} = 2\cos x = \mathcal{F}\phi(x)$$

such that when acting with $\mathcal{F}^{-1}$ the function (distribution) $\phi(x)$ results to $\boxed{\sum = 25}$

$$\phi(x) = \delta(x+1) + \delta(x-1).$$

5. (i) We start by differentiating the Laplace transform of $\phi(x,t)$ in t twice with respect to x $\boxed{22}$

$$\frac{d^2}{dx^2}\mathcal{L}_t\phi(x,t) = \mathcal{L}_t\phi_{xx}(x,t) = \mathcal{L}_t\phi_{tt}(x,t). \tag{6}$$

In the last equality we used the wave equation. Using the first hint gives

$$\begin{aligned}\mathcal{L}_t\phi_{tt}(x,t) &= t\mathcal{L}_t\phi_t(x,t) - \phi_t(x,0) \\ &= t^2\mathcal{L}_t\phi(x,t) - t\phi(x,0) - \phi_t(x,0)\end{aligned}$$

Using this together with the initial conditions (6) becomes a linear second order differential for $\mathcal{L}_t\phi(x,t)$

$$\frac{d^2}{dx^2}\mathcal{L}_t\phi(x,t) = t^2\mathcal{L}_t\phi(x,t) - t\cos x \quad \text{for} \quad 0 < x < \pi. \tag{7}$$

This may be solved by

$$\mathcal{L}_t\phi(x,t) = A(t)\sin x + B(t)\cos x. \tag{8}$$

From $\phi(\pi/2,t) = 0$ follows $A(t) = 0$. Substituting (8) into (7) then gives

$$-B(t)\cos x = t^2 B(t)\cos x - t\cos x \quad \Rightarrow \quad B(t) = \frac{t}{t^2+1}.$$

Therefore

$$\mathcal{L}_t\phi(x,t) = \frac{t}{t^2+1}\cos x$$

and hence

$$\phi(x,t) = \cos x \mathcal{L}_t^{-1}\left(\frac{t}{t^2+1}\right) = \cos x \cos t.$$

We used the second hint with $\lambda = 1$.

(ii) We compute $\boxed{3}$

$$\begin{aligned}\phi_x(x,t) &= -\sin x\cos t, & \phi_{xx}(x,t) &= -\cos x\cos t, \\ \phi_t(x,t) &= -\cos x\sin t, & \phi_{tt}(x,t) &= -\cos x\cos t.\end{aligned}$$

Thus

$$\phi_{xx}(x,t) = \phi_{tt}(x,t)$$

Clearly

$$\begin{aligned}\phi(x,0) &= \cos x, \\ \phi_t(x,0) &= -\cos x\sin 0 = 0, \\ \phi(\pi/2,t) &= \cos\pi/2\cos t = 0.\end{aligned}$$

$\boxed{\sum = 25}$