MA3605

CITY UNIVERSITY

London

BSc Honours Degree in Mathematical Science
Mathematical Science with Statistics
Mathematical Science with Computer Science
Mathematical Science with Finance and Economics
Mathematics and Finance

PART III

Mathematical Methods II

2011

Time allowed: 2 hours

Full marks may be obtained for correct answers to THREE of the five questions

Each question carries 25 marks.

1) (i) (3 marks) The linear fractional transformation is defined as

$$w = T(z) = \frac{az+b}{cz+d} \qquad \text{for } ad-bc \neq 0; a,b,c,d \in \mathbb{C}.$$

Why is the restriction $ad - bc \neq 0$ needed? For which choices of the constants a, b, c, d does this map reduce to a translation by an amount 2, a rotation of θ, a scaling by a factor -3 and the inversion map?

(ii) (6 marks) Given the two linear fractional transformations

$$T_1(z) = \frac{z-2}{2z-i} \qquad \text{and} \qquad T_2(z) = \frac{2z-i}{3iz-2}$$

compute the linear fractional transformation equivalent to the composition

$$T_1 \circ T_2(z).$$

Why is your answer expected to be a linear fractional transformation? Decompose your result into a succession of rotations, translations and inversions.

(iii) (11 marks) Construct a conformal transformation that maps a circle centered at $z = 3 + 3i$ with radius $r = 3$ to the line passing through the points $w = i$ and $w = -1$. Determine also the map that maps the line to the circle.

(iv) (5 marks) Determine the image of the circle with radius 2 through the points $z_1 = -2i$, $z_2 = -2$ and $z_3 = -4 - 2i$ which is mapped by

$$w = f(z) = \frac{(4+4i)z + (8+16i)}{z + (2+2i)}$$

into the w-plane.

2) (i) (12 marks) Find a domain on which the function

$$g_1(z) = \ln\left(\frac{z-1}{z^2-9}\right)$$

is single valued and analytic. Provide two alternative constructions: a) Take the principal branch cut for $\ln(z)$ and b) take the branch cut for $\ln(z)$ to be $\mathbb{R}^+$.

(ii) (13 marks) Using the principal branch cut for $\ln(z)$ show that

$$g_2(z) = \operatorname{arcsinh} z$$

is single valued and analytic on $\mathbb{C}\backslash\{(-i\infty, -i), (i, i\infty)\}$.

3) (i) (5 marks) Define the exponential growth of a function $f(x)$ and subsequently define the Laplace tranform $\mathcal{L}f(x)$ for the function $f(x)$.

(ii) (20 marks) Compute the inverse Laplace transform $\mathcal{L}^{-1}v(x)$ for the function

$$v(x) = \frac{5}{x^2-25},$$

by evaluating the Bromwich integral

$$\mathcal{L}^{-1}v(x) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} v(t)e^{tx}dt \quad \text{for } \gamma > \alpha,$$

where α is the exponential growth of $\mathcal{L}^{-1}v(x)$.

4) The Fourier transform $\mathcal{F}u(x) = \hat{u}(x)$ of a piecewise smooth and absolutely integrable function $u(x)$ on the real line is defined as

$$\mathcal{F}u(x) := \hat{u}(x) = \int_{-\infty}^{\infty} u(t)e^{-itx}dt.$$

(i) (7 marks) Define what is meant by the convolution of two functions $w(x)$ and $v(x)$. Compute the convolution for the two identical functions

$$w(x) = v(x) = \begin{cases} 1 & \text{for } |x| < \mu \\ 0 & \text{for } |x| > \mu \end{cases} .$$

(ii) (10 marks) Show that the Fourier transform of the convolution of two functions $v(x)$ and $u(x)$ equals the product of the Fourier transforms of $v(x)$ and $u(x)$. Verify this explicitly for the function $v(x)$ and $u(x)$ from (i).

(iii) (8 marks) Use the relation between the Fourier transform and the Fourier transform of its derivative $\mathcal{F}u'(x) = ix\mathcal{F}u(x)$ together with $\int_{-\infty}^{\infty} e^{-t^2}e^{-itx}dt = \sqrt{\pi}e^{-x^2/4}$ to compute the Fourier transform $\mathcal{F}u(x)$ of the function

$$u(x) = H_3(x)e^{-x^2},$$

where $H_3(x)$ is a Hermite polynomial of order three. The Hermite polynomial of order n is generated from

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right).$$

5) The damped harmonic oscillator for a point particle of mass m fixed on a spring with spring constant κ with external driving force $F(t)$ and damping λ is described by the equation

$$m\ddot{x}(t) + \lambda\dot{x}(t) + \kappa x(t) = F(t). \tag{1}$$

Here x is the vertical displacement of the particle as a function of time t and as usual we denote $\dot{x} \equiv dx/dt$.

(i) (11 marks) Show, using Laplace transforms, that the solution to equation (1) subject to the initial conditions $x(0) = \dot{x}(0) = 0$ is given by

$$x(t) = \frac{1}{m\omega}\int_0^\infty ds\ F(t-s)e^{-\mu s}\sin\omega s.$$

Hint: You may use

$$\begin{aligned}
\mathcal{L}u^{(n)}(x) &= x^n\mathcal{L}u(x) - \sum\nolimits_{k=0}^{n-1} x^{n-k-1}u^{(k)}(0), \\
\mathcal{L}u(x) &= \frac{\omega}{(x-\mu)^2+\omega^2} \qquad \text{for} \qquad u(x) = e^{\mu x}\sin\omega x, \\
\mathcal{L}u(x) &= \frac{x-\mu}{(x-\mu)^2+\omega^2} \qquad \text{for} \qquad u(x) = e^{\mu t}\cos\omega t,
\end{aligned}$$

(ii) (2 marks) Specify now the external force to be a kick at $t = 0$. Denoting by p the momentum transfer of this kick the external force can be represented as

$$F(t) = p\delta(t).$$

Compute the integral left in (i).

(iii) (12 marks) Specify now the external force to be a continuous sin-force with frequency ω, perhaps an electric field when the mass is charged. In this case the force is realised as

$$F(t) = F_0\sin(\omega t) \quad \text{for } t \geq 0.$$

Compute the integral left in (i).

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MA3605 Mathematical Methods II

Solutions and marking scheme for exam May 2011

INSTRUCTIONS: Full marks are obtained for correct answers to three of the five questions. Each question carries 25 marks.
(All questions and answers are unseen. Definitions, theorems, the proof in 4(ii) and 2(ii) have been seen.)

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1. (i) The restriction is needed as otherwise $T'(z)=\frac{ad-bc}{(d+cz)^2}=0$, i.e. the map would just be a constant. The notations of the lecture are [3]

$$f_T^{\Delta}(z):=z+\Delta, \qquad f_R^{z_0}(z):=zz_0 \qquad \text{and} \qquad f_I(z):=\frac{1}{z},$$

where it is understood that $f_R^{z_0}$ is a simultaneous scaling and rotation. With

$$T(z)=\frac{az+b}{cz+d}$$

we therefore have

$$\begin{aligned}
T(z) &\to f_T^2(z) && \text{for } a=1,\ b=2,\ c=0,\ d=1,\\
T(z) &\to f_R^{e^{i\theta}}(z) && \text{for } a=e^{i\theta},\ b=0,\ c=0,\ d=1,\\
T(z) &\to f_R^{-3}(z) && \text{for } a=-3,\ b=0,\ c=0,\ d=1,\\
T(z) &\to f_I(z) && \text{for } a=0,\ b=1,\ c=1,\ d=0.
\end{aligned}$$

(ii) With [6]

$$T_1(z)=\frac{z-2}{2z-i} \qquad \text{and} \qquad T_2(z)=\frac{2z-i}{3iz-2}$$

we compute

$$T_1\circ T_2(z)=\frac{\frac{2z-i}{3iz-2}-2}{\frac{2(2z-i)}{3iz-2}-i}=\frac{(2-6i)z+(4-i)}{7z}.$$

The result is expected to a be a also linear fractional transformation as the set of all linear fractional transformations forms a *group*. By the closure property of the group we expect $T_2\circ T_1$ to be a linear fractional transformation.
We can decompose this as

$$\begin{aligned}
T_1\circ T_2(z) &= \left(\frac{2}{7}-\frac{6i}{7}\right)+\frac{\frac{4}{7}-\frac{i}{7}}{z}\\
&= f_T^{\frac{2-6i}{7}}\circ f_R^{\frac{4}{7}-\frac{i}{7}}\circ f_I(z).
\end{aligned}$$

(*iii*) We use the following 11

Theorem : *The linear fractional transformation $w = T(z)$ maps three distinct points z_1, z_2, z_3 uniquely into three distinct points w_1, w_2, w_3. The map is determined by the equation*

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}.$$

We select therefore three points on the circle $z_1 = 3i, z_2 = 3, z_3 = 6+3i$ and three points on the line $w_1 = -1, w_2 = i, w_3 = 1+2i$. Substituting these points into the above formula gives

$$\frac{(w-(-1))(i-(1+2i))}{(w-(1+2i))(i-(-1))} = \frac{(z-3i)(3-(6+3i))}{(z-(6+3i))(3-3i)}.$$

Solving this for w yields

$$w = T(z) = \frac{z+3}{z-(3+6i)}.$$

The map from the line to the circle is the inverse of $T(z)$. Therefore we solve

$$z = \frac{w+3}{w-(3+6i)}$$

for w, which gives

$$w = T^{-1}(z) = \frac{(3+6i)z+3}{z-1}.$$

(*iv*) We know that the linear fractional transformation $w = T(z)$ maps circles and lines always into circles and lines. Since the points in the z-plane lie on a circle the points in the w-plane must be on a line or a circle. We compute 5

$$\begin{aligned} f(-2i) &= \frac{(8+16i)-(8+8i)i}{(2+2i)-2i} = 8+4i \\ f(-2) &= 4 \\ f(-4-2i) &= 4i \end{aligned}$$

We notice that these points lie on a circle with centre $(4, 4i)$ and with radius $r = 4$, which is the unique image. $\sum = 25$

2. (*i*) The function $g_1(z)$ has three branch points at $z = 1$ and at $z = \pm 3$. For the arguments of the logarithm we can write

$$z-1 = |z-1|\, e^{i\theta_1} \qquad \text{and} \qquad z \pm 3 = |z \pm 3|\, e^{i\theta_{2/3}}$$

such that

$$g_1(z) = \ln\left(\frac{z-1}{z^2-9}\right) = \ln(z-1) - \ln(z-3) - \ln(z+3) = \ln\left|\frac{z-1}{z^2-9}\right| + i(\theta_1 - \theta_2 - \theta_3)$$

We have now various choices for the restriction on θ_1, θ_2 and θ_3 :
a) Assume the principal values for the logarithms:

$$-\pi < \theta_1, \theta_2, \theta_3 \leq \pi$$

Let us now consider the different regions on the real axis:

- $z \in (3, \infty)$: On this part of the axis there is no problem as θ_1, θ_2 and θ_3 are all continuous when crossing the axis.
- $z \in (1, 3)$: On this line segment θ_1 and θ_2 are continuous, but θ_3 jumps and therefore we require a cut.
- $z \in (-3, 1)$: When crossing this part of the axis both θ_1 and θ_3 are discontinuous. However, the relevant quantity, which is the difference $\theta_1 - \theta_2 - \theta_3$ is continuous. Above the axis we have $\theta_2 = 0$, $\theta_1 = \theta_3 = \pi$, such that $\theta_1 - \theta_2 - \theta_3 = 0$ and below the axis we have $\theta_2 = 0$, $\theta_1 = \theta_3 = -\pi$ and therefore also $\theta_1 - \theta_2 - \theta_3 = 0$. This means no cut is required on this segment.
- $z \in (-\infty, 3)$: On this line segment we have above the axis $\theta_1 = \theta_2 = \theta_3 = \pi$ such that $\theta_1 - \theta_2 - \theta_3 = -\pi$ and below the axis we have $\theta_1 = \theta_2 = \theta_3 = -\pi$ such that $\theta_1 - \theta_2 - \theta_3 = \pi$. This means the function is discontinuous and we need a branch cut to make it analytic.

Overall we only need therefore branch cut at the line segment $(-\infty, -1)$ and $(1, 3)$ in order to make the function $g_1(z)$ single valued and analytic. 6

b) Next we assume the cut for the logarithms to be at:

$$0 < \theta_1, \theta_2, \theta_3 \leq 2\pi$$

Again we consider the different regions on the real axis:

- $z \in (3, \infty)$: On this line segment we have above the axis $\theta_1 = \theta_2 = \theta_3 = 0$ such that $\theta_1 - \theta_2 - \theta_3 = 0$ and below the axis we have $\theta_1 = \theta_2 = \theta_3 = 2\pi$ such that $\theta_1 - \theta_2 - \theta_3 = -2\pi$. This means the function is discontinuous and we need a branch cut to make it analytic.
- $z \in (1, 3)$: On this line segment we have above the axis $\theta_3 = \pi$, $\theta_1 = \theta_2 = 0$, such that $\theta_1 - \theta_2 - \theta_3 = -\pi$ and below the axis we have $\theta_3 = \pi$, $\theta_1 = \theta_2 = 2\pi$ and therefore also $\theta_1 - \theta_2 - \theta_3 = -\pi$. This means no cut is required on this segment.
- $z \in (-3, 1)$: On this line segment we have above the axis $\theta_2 = 0$, $\theta_1 = \theta_3 = \pi$, such that $\theta_1 - \theta_2 - \theta_3 = 0$ and below the axis we have $\theta_2 = 2\pi$, $\theta_1 = \theta_3 = \pi$ and therefore also $\theta_1 - \theta_2 - \theta_3 = -2\pi$. This means the function is discontinuous and we need a branch cut to make it analytic.
- $z \in (-\infty, 3)$: On this part of the axis there is no problem as θ_1, θ_2 and θ_3 are all continuous when crossing the axis.

Overall we only need therefore branch cut at the line segment $(-3, 1)$ and $(3, \infty)$ in order to make the function $g_1(z)$ single valued and analytic. 6

(*ii*) First express the arcsinh in terms of ln

$$w = \sinh z = \frac{1}{2}(e^z - e^{-z}) = \frac{1}{2}(y - y^{-1}) \quad \text{with } y = e^z.$$

Therefore

$$y^2 - 2wy - 1 = 0$$

which is solved by

$$y_{1/2} = w \pm \sqrt{w^2 + 1}.$$

Therefore taking the positive square root

$$\operatorname{arcsinh}(z) = \ln\left(z + \sqrt{z^2 + 1}\right)$$

The principal branch of ln has the negative real axis, i.e. $(-\infty, 0) \equiv \mathbb{R}^-$, as branch cut. Thus we need to guarantee that

$$a)\ \ z^2 + 1 \notin \mathbb{R}^- \qquad \text{and} \qquad b)\ \ z + \exp\left[\frac{1}{2}\ln(z^2+1)\right] \notin \mathbb{R}^-$$

a) Suppose that $z^2 + 1 \in \mathbb{R}$

$$\Rightarrow (z^2+1)^* = z^2 + 1 \quad \Leftrightarrow \quad (z^*)^2 = z^2 \quad \Rightarrow \quad z = \pm z^* \quad \Rightarrow z = x, z = iy$$

for $z = x$: $x^2 + 1 \in \mathbb{R}^+ \Rightarrow$ no restrictions arises from this possibility.
for $z = iy$: $-y^2 + 1 \in \mathbb{R}^-$ for $|y| > 1 \Rightarrow$ we need to cut out $\{(-i\infty, -i), (i, i\infty)\}$.

b) Assume that

$$z + \exp\left[\frac{1}{2}\ln(z^2+1)\right] = r \in \mathbb{R}^-$$

Therefore

$$(1 + z^2) = (r - z)^2 \quad \Leftrightarrow \quad 1 + z^2 = r^2 + z^2 - 2rz \quad \Rightarrow \ z = \frac{r^2 - 1}{2r}.$$

This means $z \in \mathbb{R}^-$ only for $r \in \mathbb{R}^-$ and no further restriction results from this possibility.
The principal branch cuts of $\operatorname{arcsinh}(z)$ are therefore at $(-i\infty, -i)$ and $(i, i\infty)$. $\boxed{\sum = 13}$

3. (*i*) **Definition:** The function $f(x)$ is said to have *exponential growth* α if there exists a constant μ such that $\boxed{\sum = 25}$

$$|f(x)| \leq \mu e^{\alpha x} \qquad \text{for } x > 0, \text{ with } \alpha, \mu \in \mathbb{R}.$$

$\boxed{2}$

Definition: The *Laplace transform* $\mathcal{L}u(x)$ of a piecewise smooth function $f(x)$ with exponential growth α is defined as

$$\mathcal{L}f(x) := \int_0^\infty f(t)e^{-tx}dt \qquad \text{for } x > \alpha.$$

$\boxed{3}$

(*ii*) Compute

$$\mathcal{L}^{-1}v(x) = \frac{1}{2\pi i}\int\limits_{\gamma-i\infty}^{\gamma+i\infty} \frac{5}{t^2-25}e^{tx}dt.$$

Parameterize $z = \varepsilon + re^{i\theta}$ and compute

$$\frac{1}{2\pi i}\oint\limits_{\Gamma} \frac{5}{z^2-25}e^{zx}dz = \frac{2\pi i}{2\pi i}\operatorname*{Res}_{z_0=\pm\omega} \frac{5}{(z-5)(z+5)}e^{zx} = \frac{1}{2}e^{5x} - \frac{1}{2}e^{-5x} = \sinh 5x$$

In order to show that 8

$$\int\limits_{\gamma-i\infty}^{\gamma+i\infty} \frac{t}{t^2-25}e^{tx}dt = \oint\limits_{\Gamma} \frac{z}{z^2-25}e^{zx}dz$$

we have to guarantee that the integral over, say Γ, parameterized by $re^{i\theta}$ for θ from $\pi/2$ to $3\pi/2$ vanishes as $r \to \infty$. Compute

$$\left|\oint\limits_{\gamma} \frac{5}{z^2-25}e^{zx}dz\right| = \left|5e^{\varepsilon x}\int\limits_{\pi/2}^{3\pi/2} \frac{re^{i\theta}}{(\varepsilon+re^{i\theta})^2-25}e^{re^{i\theta}x}d\theta\right|$$

With

$$\begin{aligned} \left|re^{i\theta}\right| &= r \\ \left|e^{re^{i\theta}x}\right| &= e^{rx\cos\theta} \leq 1 \qquad \text{for } \frac{\pi}{2} \leq \theta \leq \frac{3}{2}\pi \\ \left|(\varepsilon+re^{i\theta})^2-25\right| &> r^2-25 \end{aligned}$$

follows

$$\left|e^{\varepsilon x}\int\limits_{\pi/2}^{3\pi/2} \frac{re^{i\theta}}{(\varepsilon+re^{i\theta})^2-25}e^{re^{i\theta}x}d\theta\right| < e^{\varepsilon x}\frac{r}{r^2-25} \to 0 \quad \text{for } r \to \infty.$$

$$\Rightarrow \mathcal{L}^{-1}v(x) = \sinh 5x$$

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4. (*i*) The convolution of two functions $u(x)$ and $v(x)$ is defined as $\sum = 25$

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$$u * v(x) = \int\limits_{-\infty}^{\infty} u(s)v(x-s)ds.$$

We have

$$w(s)v(x-s) = \begin{cases} 1 & \text{for } |x-s| < \mu \text{ and } |s| < \mu \\ 0 & \text{otherwise} \end{cases}$$

This means

$$\left.\begin{array}{l} x-s<\mu \quad \Rightarrow \quad x<\mu+s<2\mu \\ -\mu<x-s \quad \Rightarrow \quad s-\mu<x \quad \Rightarrow -2\mu<x \end{array}\right\} \Rightarrow |x|<2\mu$$

When $0<x<2\mu : x-s<\mu \Rightarrow x-\mu<s<\mu \Rightarrow \int_{x-\mu}^{\mu} ds = 2\mu - x.$

When $-2\mu<x<0 : -\mu<x-s \Rightarrow -\mu<s<x+\mu \Rightarrow \int_{-\mu}^{x+\mu} ds = 2\mu + x.$

Therefore

$$w*v(x) = \begin{cases} 0 & \text{for } \; x<-2\mu \\ 2\mu+x & \text{for } \; -2\mu<x<0 \\ 2\mu-x & \text{for } \; 0<x<2\mu \\ 0 & \text{for } \; x>2\mu \end{cases}.$$

(*ii*) We verify in general 10

$$\begin{aligned} \mathcal{F}(u\star v)(x) &= \int_{-\infty}^{\infty} (u\star v)(t)e^{-itx}dt = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} ds\; u(s)v(t-s)e^{-itx} \\ &= \int_{-\infty}^{\infty} ds\; u(s) \left(\int_{-\infty}^{\infty} dt\; v(t-s)e^{-itx}e^{isx} \right) e^{-isx} \\ &= \int_{-\infty}^{\infty} ds\; u(s)e^{-isx} \left(\int_{-\infty}^{\infty} dt\; v(t)e^{-itx} \right) \\ &= \mathcal{F}(u)\mathcal{F}(v). \end{aligned}$$

First compute $\mathcal{F}(w)\mathcal{F}(v)$:

$$\mathcal{F}(w) = \int_{-\mu}^{\mu} dt\; e^{-itx} = \frac{i}{x}e^{-itx}\Big|_{-\mu}^{\mu} = \frac{i}{x}\left(e^{-ix\mu}-e^{ix\mu}\right) = \frac{2}{x}\sin(\mu x).$$

Therefore

$$\mathcal{F}(w)\mathcal{F}(v) = \left[\frac{2}{x}\sin(\mu x)\right]^2 = \frac{4}{x^2}\sin^2(\mu x).$$

On the other hand

$$\begin{aligned} \mathcal{F}(w\star v)(x) &= \int_{-2\mu}^{0} (t+2\mu)e^{-itx}dt + \int_{0}^{2\mu} (2\mu-t)e^{-itx}dt \\ &= 2\mu \int_{-2\mu}^{2\mu} e^{-itx}dt + \int_{-2\mu}^{0} te^{-itx}dt - \int_{0}^{2\mu} te^{-itx}dt \\ &= \frac{4}{x}\mu\sin(2\mu x) + \left(\frac{1}{x^2}+\frac{it}{x}\right)e^{-itx}\Big|_{-2\mu}^{0} - \left(\frac{1}{x^2}+\frac{it}{x}\right)e^{-itx}\Big|_{-2\mu}^{0} \\ &= \frac{4\mu}{x}\sin(2\mu x) + \frac{2ie^{2ix\mu}\mu}{x} - \frac{e^{2ix\mu}}{x^2} + \frac{1}{x^2} - \frac{2ie^{-2ix\mu}\mu}{x} - \frac{e^{-2ix\mu}}{x^2} + \frac{1}{x^2} \\ &= \frac{4\mu}{x}\sin(2\mu x) + \frac{2ie^{2ix\mu}\mu}{x} - \frac{e^{2ix\mu}}{x^2} + \frac{1}{x^2} - \frac{2ie^{-2ix\mu}\mu}{x} - \frac{e^{-2ix\mu}}{x^2} + \frac{1}{x^2} \\ &= \frac{4\mu}{x}\sin(2\mu x) - \frac{2\cos(2x\mu)}{x^2} - \frac{4\mu\sin(2x\mu)}{x} + \frac{2}{x^2} \\ &= -\frac{2\cos^2(x\mu)}{x^2} + \frac{2\sin^2(x\mu)}{x^2} + \frac{2}{x^2} = \frac{4\sin^2(x\mu)}{x^2} \end{aligned}$$

Hence $\mathcal{F}(w)\mathcal{F}(v) = \mathcal{F}(w)\mathcal{F}(v)$.

(*iii*) First we compute the Hermite polynomial $H_3(x)$ [8]

$$\begin{aligned} H_3(x) &= (-1)^3 e^{x^2} \frac{d^3}{dx^3}\left(e^{-x^2}\right) \\ &= -e^{x^2}\frac{d^2}{dx^2}\left[-2xe^{-x^2}\right] \\ &= e^{x^2}\frac{d}{dx}\left[2e^{-x^2} - 4x^2e^{-x^2}\right] \\ &= e^{x^2}\left[-12xe^{-x^2} + 8x^3e^{-x^2}\right] \\ &= 8x^3 - 12x, \end{aligned}$$

such that

$$u(x) = \left(8x^3 - 12x\right)e^{-x^2}.$$

Next we define $v(x) = e^{-x^2}$. Then $v'(x) = -2xe^{-x^2}$, $v''(x) = (4x^2-2)e^{-x^2}$ and $v'''(x) = (12x - 8x^3)e^{-x^2}$. Therefore

$$\begin{aligned} u(x) &= \left(8x^3 - 12x\right)e^{-x^2} \\ &= -v'''(x) \end{aligned}$$

Therefore

$$\begin{aligned} \mathcal{F}u(x) &= -\mathcal{F}v'''(x) = -ix\mathcal{F}v''(x) = x^2\mathcal{F}v'(x) = ix^3\mathcal{F}v(x) \\ &= i\sqrt{\pi}x^3e^{-x^2/4}. \end{aligned}$$

$\sum = 25$

5. (*i*) Acting with the Laplace operator $\mathcal{L}$ on [11]

$$m\ddot{x}(t) + \lambda\dot{x}(t) + \kappa x(t) = F(t)$$

gives

$$m\mathcal{L}\ddot{x}(t) + \lambda\mathcal{L}\dot{x}(t) + \kappa\mathcal{L}x(t) = \mathcal{L}F(t). \tag{1}$$

Applying $\mathcal{L}u^{(n)}(x) = x^n\mathcal{L}u(x) - \sum_{k=0}^{n-1} x^{n-k-1}u^{(k)}(0)$ for $n = 1, n = 2$ we find

$$\begin{aligned} \mathcal{L}\ddot{x}(t) &= t^2\mathcal{L}x(t) - tx(0) - \dot{x}(0) \qquad \Rightarrow \qquad \mathcal{L}\ddot{x}(t) = t^2\mathcal{L}x(t) \\ \mathcal{L}\dot{x}(t) &= t\mathcal{L}x(t) - x(0) \qquad\qquad\quad \Rightarrow \qquad \mathcal{L}\dot{x}(t) = t\mathcal{L}x(t), \end{aligned}$$

We used the initial conditions $x(0) = \dot{x}(0) = 0$. Therefore we can rewrite (1) as

$$mt^2\mathcal{L}x(t) + \lambda t\mathcal{L}x(t) + \kappa\mathcal{L}x(t) = \mathcal{L}F(t),$$

which we can solve for $\mathcal{L}x(t)$

$$\mathcal{L}x(t) = \frac{\mathcal{L}F(t)}{mt^2 + \lambda t + \kappa} = \frac{\mathcal{L}F(t)}{m(t^2 + \lambda/mt + \kappa/m)} = \frac{\mathcal{L}F(t)}{m}\frac{1}{(t+\mu)^2 + \omega^2}. \tag{2}$$

Here we completed the square and abbreviated $\omega^2 = \kappa/m - \mu^2, \mu = \lambda/2m$. Using the hint

$$\mathcal{L}u(x) = \frac{\lambda}{(x-\mu)^2+\lambda^2} \qquad \text{for} \qquad u(x) = e^{\mu x}\sin\lambda x.$$

Translating this to our notation here gives

$$\mathcal{L}v(t) = \frac{\omega}{(t+\mu)^2+\omega^2} \qquad \text{for} \qquad v(t) = e^{-\mu t}\sin\omega t. \tag{3}$$

This means we can rewrite (2) as

$$\mathcal{L}x(t) = \frac{1}{m\omega}\mathcal{L}F(t)\mathcal{L}v(t) = \frac{1}{m\omega}\mathcal{L}(F*v)(t), \tag{4}$$

where we used $\mathcal{L}(u\star v)(x) = (\mathcal{L}u)(x)(\mathcal{L}v)(x)$ in the last equality. Acting now with $\mathcal{L}^{-1}$ on (4) yields the final asnwer for $x(t)$ in form of an integral representation

$$x(t) = \frac{1}{m\omega}F*v(t) = \frac{1}{m\omega}\int_0^\infty ds\ F(t-s)e^{-\mu s}\sin\omega s. \tag{5}$$

(*ii*) When we specify $F(t) = p\delta(t)$ the solution (5) becomes **2**

$$x(t) = \frac{p}{m\omega}\int_0^\infty ds\ \delta(t-s)e^{-\mu s}\sin\omega s = \frac{p}{m\omega}e^{-\mu t}\sin\omega t.$$

(*iii*) Next we specify $F(t) = F_0\sin(\hat{\omega}t)$. We could try to evaluate **12**

$$x(t) = \frac{F_0}{m\omega}\int_0^\infty ds\ \sin[\hat{\omega}(t-s)]e^{-\mu s}\sin\omega s,$$

but it is more convenient to go back to equation (2). With the hint we have that

$$\mathcal{L}F(t) = F_0\frac{\hat{\omega}}{t^2+\hat{\omega}^2},$$

such that we obtain from equation (2)

$$x(t) = \frac{F_0\hat{\omega}}{m}\mathcal{L}^{-1}\left(\frac{1}{t^2+\hat{\omega}^2}\frac{1}{(t+\mu)^2+\omega^2}\right) = \frac{F_0\hat{\omega}}{m\chi}\mathcal{L}^{-1}\left(\frac{\alpha t+\beta}{t^2+\hat{\omega}^2}+\frac{\gamma t+\delta}{(t+\mu)^2+\omega^2}\right),$$

where

$$\chi = \hat{\omega}^4 + 2\hat{\omega}^2(\mu^2-\omega^2) + (\mu^2+\omega^2)^2 = \hat{\omega}^4 + 2\hat{\omega}^2(\frac{\lambda^2}{2m^2}-\frac{\kappa}{m}) + \frac{\kappa^2}{m^2}$$
$$\alpha = -2\mu, \quad \beta = \omega^2 - \hat{\omega}^2 + \mu^2, \quad \gamma = 2\mu, \quad \delta = \hat{\omega}^2 - \omega^2 + 3\mu^2.$$

We may now use

$$\mathcal{L}^{-1}\left(\frac{\omega}{(t-\mu)^2+\omega^2}\right) = e^{\mu t}\sin\omega t \quad \text{and} \quad \mathcal{L}^{-1}\left(\frac{t-\mu}{(t-\mu)^2+\omega^2}\right) = e^{-\mu t}\cos\omega t$$

to derive

$$x(t) = \frac{F_0\hat{\omega}}{m\chi}\left(\alpha\cos\hat{\omega}t + \frac{\beta}{\hat{\omega}}\sin\hat{\omega}t + \gamma e^{\mu t}\cos\omega t + \frac{\delta+\mu\gamma}{\omega}e^{\mu t}\sin\omega t\right).$$

Up to here will be fine. We can carry on and simplify the solution further. Depending on the sign of λ, the last two terms will lead to exponentially increasing or decreasing functions. Discarding this solution a common trick for the above type of expressions is to introduce a further parameter φ as

$$\tan\varphi = -\frac{\hat{\omega}\alpha}{\beta} = \frac{\lambda\hat{\omega}}{k - m\hat{\omega}^2}$$

such that

$$x(t) = \frac{F_0\hat{\omega}}{m\chi}\sin(\hat{\omega}t - \varphi).$$

$\boxed{\sum = 25}$