MA3603

CITY UNIVERSITY
London

BSc Honours Degree in Mathematical Science
Mathematical Science with Statistics
Mathematical Science with Computer Science
Mathematical Science with Finance and Economics
Mathematics and Finance

PART III

Mathematical Methods II

2012

Time allowed: 3 hours

Full marks may be obtained for correct answers to FOUR of the six questions, with TWO questions selected from each section.

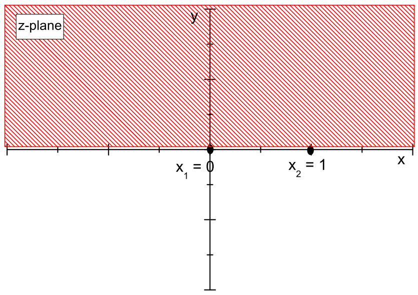
In each section, if more than TWO questions are answered, the best TWO marks will be credited.

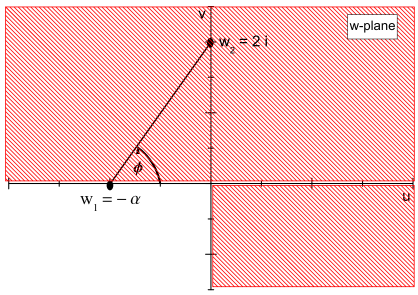
Each question carries 25 marks.

SECTION A

1) (*i*) (3 marks) State the Schwarz-Christoffel theorem which characterises the mapping of the upper half plane $\operatorname{Im} z > 0$ into an n-sided polygon. How is the theorem modified when the upper half plane is mapped onto an unbounded region?

(*ii*) (10 marks) Determine the Schwarz-Christoffel transformation $f : z \mapsto w$, which maps the upper half plane onto the region as indicated in the figure:





Hint: At first map $x_1 = 0, x_2 = 1$ onto $w_1 = -\alpha$, $w_2 = 2i$ and then let $\alpha \to 0$.

(*iii*) (6 marks) Construct a linear fractional transformation that maps the interior of the unit circle to the exterior of the circle with radius 2 centred at $(2, -1)$.

(*iv*) (6 marks) Prove the following theorem: *A harmonic function $\psi(x, y)$ transforms into a harmonic function $\psi(u, v)$ when changing variables as $z = x + iy = f(w) = f(u + iv)$ with f being an analytic function.* How is this theorem used in potential theory?

2) (*i*) (3 marks) Define the branch, the branch cut and the branch point of a multi-valued function.

(*ii*) (14 marks) Find the largest domain on which the function

$$g(z) = \ln\left(\frac{z^2 - 25}{z - 8}\right)$$

is single valued and analytic. Provide two alternative constructions: a) Take the principal branch cut for the logarithmic function and b) take the branch cut for the logarithmic function to be at $\mathbb{R}^+$.

(*iii*) (8 marks) Compute specific values of

$$(a)\ \ln\left(\sqrt{12} + 2i\right), \qquad (b)\ \arccos\left[\ln(-1)\right], \qquad (c)\ \operatorname{arctanh}(-\pi).$$

In (a) and (b) compute the principal values and in (c) use the normalization $\operatorname{arctanh}(0) = i\pi$ to select a branch. Provide explicit arguments for any choice of signs in your calculations.

3) (*i*) (2 marks) Explain in which sense a Laplace transform may by obtained from a Fourier transform by a variable transformation.

(*ii*) (4 marks) Explain what is meant by exponential growth of a function and provide the explicit argument of how this is used to ensure the existence of a Laplace transform.

(*iii*) (10 marks) Employ the Laplace transforms to solve the second order differential equation

$$y''(x) + 2y'(x) + y(x) = x$$

with intial conditions $y(0) = -1$ and $y'(0) = 0$.

(*iv*) (9 marks) Employ the convolution theorem to solve the integral equation

$$\phi(t) = 2\cos t - \int_0^t (t - s)\phi(s)ds.$$

You may use the following table in any part of this question:

$u(x)$	$\mathcal{L}u(x)$
1	$\frac{1}{x}$
x	$\frac{1}{x^2}$
$\cos x$	$\frac{x}{1+x^2}$
$\sin x$	$\frac{1}{1+x^2}$
$x\cos x$	$\frac{x^2-1}{(1+x^2)^2}$
$x\sin x$	$\frac{2x}{(1+x^2)^2}$
$\exp(-x)$	$\frac{1}{1+x}$
$v'(x)$	$x\mathcal{L}v(x) - v(0)$

SECTION B

4) (a) Consider the nonhomogeneous boundary value problem

$$L[y] \equiv \frac{d}{dx}\left[\frac{1}{2x^2}\frac{dy}{dx}\right] + \frac{1}{x^4}y = -f(x), \qquad 1 < x < 2,$$

$$y'(1) = 0, \qquad y(2) - y'(2) = 0.$$

(i) (4 marks) Show that $y_1(x) = x$ and $y_2(x) = x^2$ form a fundamental set of solutions to the associated homogeneous equation $L[y] = 0$.

(ii) (11 marks) State the properties of the Green's function for a self-adjoint equation with homogeneous boundary conditions. Use these properties to derive the Green's function for the problem under consideration. (You may also use the symmetry property.)

(iii) (4 marks) Use the Green's function to express the solution of the boundary value problem as a sum of two integrals. Hence evaluate the solution when $f(x) = \frac{1}{x^2}$.

(b) (6 marks) State Picard's existence and uniqueness theorem for the initial value problem

$$\frac{dy}{dx} = f(x, y), \qquad y(x_0) = \eta_0.$$

Use the theorem to prove that the initial value problem

$$y\frac{dy}{dx} = \cos(xy), \qquad y(0) = 10,$$

has a unique solution on the interval $0 \leq x \leq 25$.

5) (a) Consider the partial differential equation

$$u_{xx} + 5u_{xy} + 4u_{yy} = 18y - 18x.$$

(i) (4 marks) Classify the equation as hyperbolic, parabolic or elliptic. Find the equations of its characteristics.

(ii) (6 marks) By transforming to the coordinates $\xi = 4x - y$ and $\eta = x - y$, show that the canonical form of the equation is

$$\overline{u}_{\xi\eta} = 2\eta,$$

where $\overline{u}(\xi, \eta) = u(x(\xi, \eta), y(\xi, \eta))$.

(iii) (3 marks) Hence find the general solution $u(x, y)$ of the equation.

(b) Consider the system of differential equations

$$\begin{aligned} y_1'(x) &= 5y_1(x) - 4y_2(x) \\ y_2'(x) &= 3y_1(x) - 3y_2(x) \end{aligned}$$

(i) (1 mark) Write the system in the matrix form $\mathbf{y}'(x) = A\mathbf{y}(x)$.

(ii) (5 marks) Determine the eigenvalues and eigenvectors of the matrix A.

(iii) (1 mark) Hence obtain the general solution of the system.

(iv) (5 marks) Write down a fundamental matrix for the system and determine the corresponding transition matrix.

6) (*a*) (11 marks) Find the eigenvalues and eigenfunctions of the boundary-value problem

$$\frac{d^2y}{dx^2} + \lambda y = 0, \qquad 0 < x < 1,$$
$$y'(0) = 0, \quad y(1) = 0.$$

(*b*) Consider the nonhomogeneous initial-boundary-value problem

$$\begin{aligned} &u_t - u_{xx} = h(x,t), && 0 < x < 1, \quad t > 0, \\ &u_x(0,t) = 0, \quad u(1,t) = 0, && t > 0, \\ &u(x,0) = f(x), && 0 < x < 1. \end{aligned}$$

(*i*) (4 marks) By using the method of separation of the variables on the associated homogeneous problem, show that the eigenfunctions determined in (a) give a basis for an eigenfunction expansion of the solution of the nonhomogeneous problem.

(*ii*) (5 marks) Show that a formal solution of the nonhomogeneous problem is given by

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t) X_n(t),$$

where the $X_n(x)$ are the eigenfunctions obtained in (a), the $b_n(t)$ satisfy

$$\begin{aligned} &b_n'(t) + \left(n - \frac{1}{2}\right)^2 \pi^2 b_n(t) = \gamma_n(t), && n = 1, 2, \ldots, \\ &b_n(0) = \beta_n, && n = 1, 2, \ldots, \end{aligned}$$

and $\gamma_n(t)$ and β_n are the coefficients of $X_n(x)$ in the expansions of $h(x,t)$ and $f(x)$ respectively.

(*iii*) (5 marks) By solving the differential equation for $b_n(t)$, show that

$$b_n(t) = e^{-\lambda_n t}\beta_n + \int_0^t e^{-\lambda_n (t-\tau)} \gamma_n(\tau)\, d\tau,$$

where $\lambda_n = \left(n - \frac{1}{2}\right)^2 \pi^2$.

Internal Examiners:	Professor J. Chuang
	Professor A. Fring
External Examiners:	Professor J. Rickard
	Professor J. Lamb

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1) (*i*) (3 marks) *(Schwarz-Christoffel theorem) Given an n-sided polygon with vertices w_i and exterior angles $\theta_i = \mu_i \pi$ for $1 \leq i \leq n$. Then there exist always n real numbers x_i for $1 \leq i \leq n$ together with a complex constant $c \in \mathbb{C}$ and an analytic function $f : z \mapsto w$ whose derivative is given by*

$$f'(z) = c \prod_{i=1}^{n-1} (z - x_i)^{-\mu_i} \qquad c \in \mathbb{C}, -1 < \mu_i < 1,$$

which maps the upper half plane one-to-one onto the interior of the polygon. The points are mapped as $w_i = f(x_i)$ for $1 \leq i \leq n-1$ and $w_n = \lim_{x \to \pm\infty} f(x)$.

The expression for $f'(z)$ remains the same when the target space is an unbounded region, but in that case we have $\theta_1 + \theta_2 + \theta_3 + \ldots + \theta_{n-1} \leq \pi$ and $\theta_n > \pi$.

(*ii*) (10 marks) Initially we have

$$f'(z) = c(z + \alpha)^{-\phi/\pi}(z-1)^{(\pi/2+\phi)/\pi}$$

with ϕ being the angle as indicated in the figure. Letting $\alpha \to 0$ the angle becomes $\phi = \pi/2$, such that

$$f'(z) = c(z)^{-1/2}(z-1).$$

Integrating we obtain

$$\begin{aligned} f(z) &= c \int (\sqrt{z} - z^{-1/2}) dz \\ &= c\frac{2}{3} z^{3/2} - 2c\sqrt{z} + \tilde{c} = c\frac{2}{3}\sqrt{z}(z-3) + \tilde{c}. \end{aligned}$$

We also have

$$\begin{aligned} f(x_1) &= w_1 \quad \Leftrightarrow \quad f(0) = 0 \quad \Leftrightarrow \quad \tilde{c} = 0, \\ f(x_2) &= w_2 \quad \Leftrightarrow \quad f(1) = 2i \quad \Leftrightarrow \quad c\frac{2}{3}\sqrt{1}(1-3) = 2i, \end{aligned}$$

which fixes $c = -3i/2$ and hence

$$f(z) = -i\sqrt{z}(z-3).$$

(*iii*) (6 marks) One can select three distinct points on each circle, i.e. z_1, z_2, z_3 and w_1, w_2, w_3 and subsequently solve

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)},$$

for w. It is easier to decompose the linear fractional transformation. We know that the inverse map $f_I(z)$ maps the interior of the unit circle to its exterior, then we use a scaling ("rotation") $f_R^2(z)$ to increase the radius from 1 to 2 and finally we translate the centre from the origin to $(2,-1)$ by a translation map $f_T^{2-i}(z)$. Therefore

$$\begin{aligned} f(z) &= f_T^{2-i} \circ f_R^2 \circ f_I(z) = f_T^{2-i} \circ f_R^2 \left(\frac{1}{z}\right) = f_T^{2-i}\left(\frac{2}{z}\right) = \frac{2}{z} + 2 - i \\ &= \frac{(2-i)z+2}{z}. \end{aligned}$$

This is clearly a linear fractional transformation, i.e. of the form

$$w = T(z) = \frac{az+b}{cz+d} \qquad \text{for } ad-bc \neq 0; a,b,c,d \in \mathbb{C}.$$

(*iv*) (6 marks)

Proof : Take $\psi(x,y)$ to be a harmonic function and $z = f(w)$ to be an analytic function.
$\Rightarrow \exists$ a conjugate function $\tilde{\psi}(x,y)$, i.e. a harmonic function such that $\psi(x,y)$, $\tilde{\psi}(x,y)$ satisfy the Cauchy-Riemann equations.
$\Rightarrow$ The newly defined function $\phi(x,y) = \psi(x,y) + i\tilde{\psi}(x,y)$ is an analytic function of z, which follows by the corollary: *The real and imaginary parts of an analytic function are harmonic functions. Conversely, if the two functions $u(x,y)$ and $v(x,y)$ are harmonic functions then $f(z) = u(x,y) + iv(x,y)$ is an analytic function.*
$\Rightarrow \phi(z) = \phi(f(w))$ is an analytic function of w, since an analytic function of an analytic function is an analytic function.
$\Rightarrow \psi$ is a harmonic function of u, v.$\square$

In potential theory this guarantees that in a boundary value problem the Laplace equation is preserved, such that we can map complicated domains to easier ones.

2) (*i*) (3 marks)

Definition: *A* <u>branch</u> *F(z) of a multi-valued function $f(z)$ is any single valued function, which is analytic in some domain $D \subset \mathbb{C}$, where F(z_0) =$f(z_0)$ for all $z_0 \in D$.*

Definition: *A* branch cut *is a curve in the complex plane across which an analytic multivalued function is discontinuous.*

Definition: *A point which is shared by all branches of the function is called a* branch point.

(*ii*) (14 marks) The function $g(z)$ has three branch points at $z = 8$ and at $z = \pm 5$. For the arguments of the logarithm we can write

$$z \pm 5 = |z \pm 5|\, e^{i\theta_{2/1}} \qquad \text{and} \qquad z - 8 = |z - 8|\, e^{i\theta_3}$$

such that

$$\begin{aligned} g(z) &= \ln\left(\frac{z^2 - 25}{z - 8}\right) = \ln(z + 5) + \ln(z - 5) - \ln(z - 8) \\ &= \ln\left|\frac{z^2 - 25}{z - 8}\right| + i(\theta_1 + \theta_2 - \theta_3) \end{aligned}$$

We have now various choices for the restriction on θ_1, θ_2 and θ_3 :

a) Assume the principal values for the logarithms: [7]

$$-\pi < \theta_1, \theta_2, \theta_3 \leq \pi$$

Let us now consider the different regions on the real axis:

- $z \in (8, \infty)$: On this part of the axis there is no problem as θ_1, θ_2 and θ_3 are all continuous when crossing the axis.
- $z \in (5, 8)$: On this line segment θ_1 and θ_2 are continuous, but θ_3 jumps and therefore we require a cut.
- $z \in (-5, 5)$: When crossing this part of the axis both θ_2 and θ_3 are discontinuous. However, the relevant quantity, which is the difference $\theta_1 + \theta_2 - \theta_3$ is continuous. Above the axis we have $\theta_1 = 0$, $\theta_2 = \theta_3 = \pi$, such that $\theta_1 + \theta_2 - \theta_3 = 0$ and below the axis we have $\theta_1 = 0$, $\theta_2 = \theta_3 = -\pi$ and therefore also $\theta_1 + \theta_2 - \theta_3 = 0$. This means no cut is required on this segment.
- $z \in (-\infty, -5)$: On this line segment we have above the axis $\theta_1 = \theta_2 = \theta_3 = \pi$ such that $\theta_1 + \theta_2 - \theta_3 = \pi$ and below the axis we have $\theta_1 = \theta_2 = \theta_3 = -\pi$ such that $\theta_1 + \theta_2 - \theta_3 = -\pi$. This means the function is discontinuous and we need a branch cut to make it analytic.

Overall we only need therefore branch cuts at the line segment $(-\infty, -5)$ and $(5, 8)$ in order to make the function $g(z)$ single valued and analytic.

ii) Next we assume the cuts for the logarithms to be at: [7]

$$0 < \theta_1, \theta_2, \theta_3 \leq 2\pi$$

Again we consider the different regions on the real axis:

- $z \in (8, \infty)$: On this line segment we have above the axis $\theta_1 = \theta_2 = \theta_3 = 0$ such that $\theta_1 + \theta_2 - \theta_3 = 0$ and below the axis we have $\theta_1 = \theta_2 = \theta_3 = 2\pi$ such that $\theta_1 + \theta_2 - \theta_3 = 2\pi$. This means the function is discontinuous and we need a branch cut to make it analytic.
- $z \in (5, 8)$: On this line segment we have above the axis $\theta_3 = \pi$, $\theta_1 = \theta_2 = 0$, such that $\theta_1 + \theta_2 - \theta_3 = -\pi$ and below the axis we have $\theta_3 = \pi$, $\theta_1 = \theta_2 = 2\pi$ and therefore also $\theta_1 + \theta_2 - \theta_3 = 3\pi$. This means the function is discontinuous and we need a branch cut to make it analytic.
- $z \in (-5, 5)$: On this line segment we have above the axis $\theta_1 = 0$, $\theta_2 = \theta_3 = \pi$, such that $\theta_1 + \theta_2 - \theta_3 = 0$ and below the axis we have $\theta_1 = 2\pi$, $\theta_2 = \theta_3 = \pi$ and therefore we have $\theta_1 + \theta_2 - \theta_3 = 2\pi$. This means the function is discontinuous and we need a branch cut to make it analytic.
- $z \in (-\infty, -5)$: On this part of the axis there is no problem as θ_1, θ_2 and θ_3 are all continuous when crossing the axis.

Overall we need therefore a branch cut at the line segment $(-5, \infty)$ in order to make the function $g(z)$ single valued and analytic.

(iii) (8 marks) We compute

(a) [2]

$$\begin{aligned} \ln\left(2\sqrt{3} + 2i\right) &= \ln\left(4\cos\frac{\pi}{6} + 4i\sin\frac{\pi}{6}\right) \\ &= \ln(4) + \ln\left(e^{i\pi/6 + 2\pi i n}\right) \qquad \text{with } n \in \mathbb{Z} \\ &= 2\ln 2 + i\frac{\pi}{6} + 2\pi i n \end{aligned}$$

For the principal value of $\ln(z)$ we require $-\pi < \arg z \leq \pi$, such that $n = 0$.

(b) Since $\arccos(z) = w$ we have [3]

$$z = \cos w = \frac{1}{2}\left(e^{iw} + e^{-iw}\right) \quad \Rightarrow e^{2iw} - 2ze^{iw} + 1 = 0.$$

Therefore

$$e^{iw + 2\pi i n} = z \pm \sqrt{z^2 - 1}$$

such that

$$\arccos(z) = -i\ln\left(z + \sqrt{z^2 - 1}\right).$$

We take the principal values and note that $\pm\sqrt{z^2-1} = \sqrt{z^2-1}$ with $z^2-1 = e^{iw'+2\pi in}$. Now we compute $\arccos[\ln(-1)]$. Taking the principal value $\ln(-1) = i\pi$

$$\begin{aligned}
\arccos[\ln(-1)] &= -i\ln\left(i\pi + \sqrt{-\pi^2-1}\right) \\
&= -i\ln\left(i\pi + i\sqrt{\pi^2+1}\right) \\
&= -i\ln\left[\left(\pi+\sqrt{1+\pi^2}\right)e^{i\pi/2}\right] \\
&= \frac{\pi}{2} - i\ln\left(\pi+\sqrt{1+\pi^2}\right).
\end{aligned}$$

(c) Since $\operatorname{arctanh}(z) = w$ we have [3]

$$z = \tanh w = \frac{e^w - e^{-w}}{e^w + e^{-w}} \quad \Rightarrow ze^{2w} + z = e^{2w} - 1.$$

Therefore

$$e^{2w+2\pi in} = \frac{1+z}{1-z},$$

such that

$$\begin{aligned}
\operatorname{arctanh}(z) &= \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right) + i\pi n \\
&= \frac{1}{2}\ln\left|\frac{1+z}{1-z}\right| + \frac{i}{2}\arg\left(\frac{1+z}{1-z}\right) + i\pi n.
\end{aligned}$$

This means $\operatorname{arctanh}(0) = i\pi n$ and we have to select $n = 1$. We use this to evaluate

$$\begin{aligned}
\operatorname{arctanh}(-\pi) &= \frac{1}{2}\ln\left|\frac{1-\pi}{1+\pi}\right| + \frac{i}{2}\arg\left(\frac{1-\pi}{1+\pi}\right) + i\pi \\
&= \frac{1}{2}\ln\left|\frac{1-\pi}{1+\pi}\right| + \frac{3}{2}i\pi.
\end{aligned}$$

3) (i) (2 marks) *The* <u>Fourier transform</u> $\mathcal{F}$u(x)=$\hat{u}(x)$ *of a piecewise smooth and absolutely integrable function* $u(x)$ *on the real line is defined as*

$$\mathcal{F}u(x) := \hat{u}(x) = \int_{-\infty}^{\infty} u(t)e^{-itx}dt.$$

The Laplace transform of a function $u(x)$ is essentially the Fourier transform of this function whose argument is rotated by $-\pi/2$

$$\mathcal{F}u(f_R^{-\pi/2}x) = \mathcal{F}u(e^{-i\pi/2}x) = \mathcal{F}u(-ix) = \int_{-\infty}^{\infty} u(t)e^{-xt}dt.$$

Since in comparison with the Fourier transform we have now traded the oscillatory function e^{-ixt} for e^{-xt} we require a very strong decay for $u(t)$ when $t \to -\infty$. To avoid this one usually assumes that $u(x) = 0$ for $-\infty < x < 0$, such that *the* Laplace transform $\mathcal{L}$u(x) *of a piecewise smooth function* u(x) *with exponential growth* α *is defined as*

$$\mathcal{L}u(x) := \int_0^\infty u(t)e^{-tx}dt \qquad \text{for } x > \alpha.$$

(ii) (4 marks) *A function* u(x) *is said to have* exponential growth α *if there exists a constant* μ *such that*

$$|u(x)| \leq \mu e^{\alpha x} \qquad \text{for } x > 0, \text{ with } \alpha, \mu \in \mathbb{R}.$$

The existence of the integral is guaranteed by the following argument

$$\begin{aligned} \mathcal{L}u(x) &\leq \int_0^\infty |u(t)|\, e^{-tx}dt \\ &\leq \mu \int_0^\infty e^{\alpha x}e^{-tx}dt \\ &= \mu \int_0^\infty e^{(\alpha - x)t}dt < \infty \qquad \text{for } x > \alpha. \end{aligned}$$

(iii) (10 marks) Acting with $\mathcal{L}$ on the differential equation gives

$$\mathcal{L}y''(x) + 2\mathcal{L}y'(x) + \mathcal{L}y(x) = \mathcal{L}x.$$

Using the last identity from the table and the expression for $\mathcal{L}x$ gives

$$x\mathcal{L}y'(x) - y'(0) + 2x\mathcal{L}y(x) - 2y(0) + \mathcal{L}y(x) = \frac{1}{x^2}$$

which becomes

$$x^2\mathcal{L}y(x) - xy(0) - y'(0) + 2x\mathcal{L}y(x) - 2y(0) + \mathcal{L}y(x) = \frac{1}{x^2}.$$

Using the intial conditions and solving for $\mathcal{L}y(x)$ yields

$$\mathcal{L}y(x) = \frac{1}{(1+x)^2}\left(\frac{1}{x^2} - 2 - x\right) = \frac{1}{x^2} - \frac{2}{x} + \frac{1}{1+x}.$$

Acting now with the inverse Laplace transform on this equation and using the Laplace transforms provided in the table we obtain

$$y(x) = x - 2 + \exp(-x).$$

(*iv*) (9 marks) *The* <u>convolution</u> *of two functions u(x) and v(x) is defined as*

$$u \star v(x) = \int_{-\infty}^{\infty} u(s)v(x-s)ds.$$

When $u(x) = v(x) = 0$ for $x < 0$ this becomes

$$u \star v(x) = \int_0^x u(t)v(x-t)dt.$$

The convolution theorem states: *The Laplace transform of the convolution of the two functions* u *and* v*, i.e.* $u \star v(x)$ *equals the product of the Laplace transforms these functions*

$$\mathcal{L}(u \star v)(x) = (\mathcal{L}u)(x)(\mathcal{L}v)(x).$$

Acting now with $\mathcal{L}$ on the integral equation

$$\phi(t) = 2\cos t - \int_0^t (t-s)\phi(s)ds$$

gives

$$\mathcal{L}\phi(t) = 2\mathcal{L}v(t) - \mathcal{L}u(t)\mathcal{L}\phi(t).$$

where $v(t) = \cos t$ and $u(t) = t$. Solving this for $\mathcal{L}\phi(t)$ gives

$$\mathcal{L}\phi(t) = \frac{2\mathcal{L}v(t)}{1 + \mathcal{L}u(t)}.$$

Using the Laplace transforms provided in the table we obtain

$$\mathcal{L}\phi(t) = \frac{2t^3}{(1+t^2)^2} = \frac{2t}{1+t^2} - \frac{2t}{(1+t^2)^2}.$$

Taking the inverse Laplace tranform $\mathcal{L}^{-1}$ of this equation yields together with the table

$$\phi(t) = 2\cos t - t\sin t.$$

4) (a) (i) [4] We have $L[x] = -1/x^3 + 1/x^3 = 0$ and $L[x^2] = -1/x^2 + 1/x^2 = 0$. The Wronskian

$$W(y_1, y_2)(x) = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = x^2 \neq 0$$

for $1 < x < 2$. So y_1 and y_2 form a fundamental set of solutions.

(ii) [11] The Green's function for the self-adjoint equation $(py')' + qy = -f$ has the following properties:

(1) $G(x,t)$ satisfies the homogeneous DE with respect to x ($x \neq t$)

(2) $G(x,t)$ satisfies the boundary conditions

(3) $G(x,t)$ is continuous at $x = t$

(4) $\frac{\partial G}{\partial x}$ is discontinuous at $x = t$ with $\frac{\partial G}{\partial x}|_{x=t+0} - \frac{\partial G}{\partial x}|_{x=t-0} = -\frac{1}{p(t)}$

The equation as given is already in self-adjoint form, with $p(x) = 1/2x^2$. Using (1) we have

$$G(x,t) = \begin{cases} A(t)x + B(t)x^2, & 1 \le x \le t, \\ C(t)x + D(t)x^2, & t \le x \le 2. \end{cases}$$

Then by (2) $G_x(1,t) = A + 2B = 0 \implies A = -2B$, and $G(2,t) - G_x(2,t) = C = 0$. Thus

$$G(x,t) = \begin{cases} B(t)x(x-2), & 1 \le x \le t, \\ D(t)x^2, & t \le x \le 2. \end{cases}$$

Then by the symmetry property

$$G(x,t) = \begin{cases} Kx(x-2)t^2, & 1 \le x \le t, \\ Kx^2 t(t-2), & t \le x \le 2, \end{cases}$$

where K is some constant. Then (3) is already satisfied, and (4) gives $K(2x)t(t-2)|_{x=t} - K(2x-2)t^2|_{x=t} = -(1/2t^2)^{-1} = -2t^2$, which implies $K = 1$. So

$$G(x,t) = \begin{cases} x(x-2)t^2, & 1 \le x \le t, \\ x^2 t(t-2), & t \le x \le 2, \end{cases}$$

(iii) [4] We have

$$y(x) = \int_1^2 G(x,t)f(t)dt = \int_1^x x^2 t(t-2)f(t)dt + \int_x^2 x(x-2)t^2 f(t)dt.$$

When $f(x) = \frac{1}{x^2}$,

$$\begin{aligned} y(x) &= x^2 \int_1^x (1 - \frac{2}{t})dt + x(x-2) \int_x^2 dt \\ &= x^2(x - 2\ln x - 1) + x(x-2)(2-x) \\ &= x(3x - 4 - 2x\ln x). \end{aligned}$$

(b) [6] Picard's Theorem: If $f(x, y)$ and f_y are continuous in the rectangle $R = \{(x, y) : x_0 \le x \le x + a, \quad |y - \eta_0| \le b\}$ then the initial-value problem has a unique solution in the interval $x_0 \le x \le x_0 + \alpha$, where $\alpha = \min(a, b/M)$ and $M = \max_{(x,y)\in R} |f(x, y)|$.
For the particular IVP in question,

$$f(x, y) = \cos(xy)/y, \quad f_y = \frac{-\sin(xy)xy - \cos(xy)}{y^2}.$$

So f and f_y are continuous for all x and $y \ne 0$, and therefore f and f_y are continuous in the rectangle $R = \{(x, y) : 0 \le x \le a, \quad |y - 10| \le b\}$ for any $a > 0$ and $10 > b > 0$. Now $M = \max_R |f(x, y)| = f(0, 10 - b) = \frac{1}{10-b}$, since $|\cos(xy)| \le 1$ and $|y| = y \ge 10 - b$. Hence $\alpha = \min\{a, b(10 - b)\}$. Taking $a = 25$ and $b = 5$ we have $\alpha = 25$, and so there is a unique solution for $0 < x < 25$.

5) (a) (i) [4] Since $b^2 - 4ac = 5^2 - 4(1)(4) = 9 > 0$, the equation is hyperbolic. The characteristics are solutions of the differential equation

$$\left(\frac{dy}{dx}\right)^2 - 5\frac{dy}{dx} + 4 = 0.$$

So $dy/dx = 4, 1$ giving characteristics $y = 4x + c_1$ and $y = x + c_2$.

(ii) [6] The characteristic coordinates are thus $\xi = -c_1 = 4x - y$ and $\eta = -c_2 = x - y$. To transform to canonical form, we calculate

$$\begin{aligned} u_x &= 4\overline{u}_\xi + \overline{u}_\eta \\ u_y &= -\overline{u}_\xi - \overline{u}_\eta \\ u_{xx} &= (4\partial_\xi + \partial_\eta)(4\overline{u}_\xi + \overline{u}_\eta) = 16\overline{u}_{\xi\xi} + 8\overline{u}_{\xi\eta} + \overline{u}_{\eta\eta} \\ u_{xy} &= (4\partial_\xi + \partial_\eta)(-\overline{u}_\xi - \overline{u}_\eta) = -4\overline{u}_{\xi\xi} - 5\overline{u}_{\xi\eta} - \overline{u}_{\eta\eta} \\ u_{yy} &= (-\partial_\xi - \partial_\eta)(-\overline{u}_\xi - \overline{u}_\eta) = \overline{u}_{\xi\xi} + 2\overline{u}_{\xi\eta} + \overline{u}_{\eta\eta} \end{aligned}$$

So the PDE transforms to

$$(16\overline{u}_{\xi\xi} + 8\overline{u}_{\xi\eta} + \overline{u}_{\eta\eta}) + 5(-4\overline{u}_{\xi\xi} - 5\overline{u}_{\xi\eta} - \overline{u}_{\eta\eta}) + 4(\overline{u}_{\xi\xi} + 2\overline{u}_{\xi\eta} + \overline{u}_{\eta\eta}) = -18\eta.$$

So $-9\overline{u}_{\xi\eta} = -18\eta \implies \overline{u}_{\xi\eta} = 2\eta$.

(iii) [3] Integrating the DE with respect to η we have $\overline{u}_\xi = \eta^2 + f(\xi)$, and then with respect to ξ we obtain $\overline{u} = \xi\eta^2 + F(\xi) + G(\eta)$. So the general solution is

$$u(x,y) = (4x-y)(x-y)^2 + F(4x-y) + G(x-y),$$

where F and G are arbitrary.

(b) (i) [1] In this form

$$\mathbf{y} = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} \qquad \text{and} \qquad A = \begin{pmatrix} 5 & -4 \\ 3 & -3 \end{pmatrix}.$$

(ii) [5] The characteristic polynomial is $(5-\lambda)(-3-\lambda)-(-12) = \lambda^2 - 2\lambda - 3 = (\lambda+1)(\lambda-3)$. So the eigenvalues are -1 and 3 and the corresponding eigenvectors are $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

(iii) [1] Thus the general solution is

$$y(x) = Ae^{-x}\begin{pmatrix} 2 \\ 3 \end{pmatrix} + Be^{3x}\begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

(iv) [5] A fundamental matrix is

$$M(x) = \begin{pmatrix} 2e^{-x} & 2e^{3x} \\ 3e^{-x} & e^{3x} \end{pmatrix}.$$

Then

$$[M(x)]^{-1} = \frac{1}{-4e^{2x}}\begin{pmatrix} e^{3x} & -2e^{3x} \\ -3e^{-x} & 2e^{-x} \end{pmatrix} = \frac{1}{4}\begin{pmatrix} -e^{x} & 2e^{x} \\ 3e^{-3x} & -2e^{-3x} \end{pmatrix}$$

and the transition matrix is

$$M(x,x_0) = M(x)[M(x_0)]^{-1} = \frac{1}{4}\begin{pmatrix} 2e^{-x} & 2e^{3x} \\ 3e^{-x} & e^{3x} \end{pmatrix}\begin{pmatrix} -e^{x_0} & 2e^{x_0} \\ 3e^{-3x_0} & -2e^{-3x_0} \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} -2e^{-(x-x_0)} + 6e^{3(x-x_0)} & 4e^{-(x-x_0)} - 4e^{3(x-x_0)} \\ -3e^{-(x-x_0)} + 3e^{3(x-x_0)} & 6e^{-(x-x_0)} - 2e^{3(x-x_0)} \end{pmatrix}.$$

6) (a) [11] This is a S-L problem with $p(x) = 1$, $q(x) = 0$ and $r(x) = 1$, so the eigenvalues are real.

Suppose $\lambda < 0$, and put $\lambda = -\mu^2$, where μ is positive, real. Then the DE becomes $y'' - \mu^2 y = 0$, which has general solution $y(x) = A\cosh\mu x + B\sinh\mu x$. The first BC implies $B = 0$, and then the second that $y(1) = A\cosh\mu = 0 \implies A = 0$, since $\cosh\mu \neq 0$. So $y(x) = 0$ and there are no nontrivial solutions in this case.

Suppose $\lambda = 0$. Then the DE is $y''(x) = 0$ which has general solution $y(x) = Ax + B$. The first BC gives $A = 0$ and then the second that $B = 0$. So again thre are no nontrivial solutions.

Suppose $\lambda > 0$, and let $\lambda = \mu^2$, where μ is positive, real. Then the DE is $y'' + \mu^2 y = 0$, which has GS $y(x) = A\cos\mu x + B\sin\mu x$. The first BC gives $B = 0$, and the second $y(1) = A\cos\mu = 0$. So either $A = 0$, which gives the trivial solution, or $\cos\mu = 0$, which implies that $\mu = \left(n - \frac{1}{2}\right)\pi$, $n = 1, 2, \ldots$.

So the e-values are $\lambda_n = \left(n - \frac{1}{2}\right)^2 \pi^2$ and the e-functions are $y_n(x) = \cos\left(n - \frac{1}{2}\right)\pi x$, $n = 1, 2, \ldots$.

(b) i. [4] The associated problem is

$$u_t - u_{xx}, \quad u(0, y) = 0, u_x(0, t) = 0.$$

Let $u(x, t) = X(x)T(t)$. Then the DE implies $XT' - X''T = 0$ which implies that $\frac{T'}{T} = \frac{X''}{X} = -\lambda$. The BCs implies $u_x(0, t) = X'(0)T(t) = 0 \implies X'(0) = 0$ and $u(1, t) = X(1)T(t) = 0 \implies X(1) = 0$. So a basis is given by the eigenfunctions of $X'' + \lambda X = 0, \quad X'(0) = 0, \quad X(1) = 0$.

ii. [5] If $u(x, t) = \sum_{n=1}^{\infty} b_n(t)X_n(x)$, with $X_n(x) = \cos\left(n - \frac{1}{2}\right)\pi x$ then the DE implies $\sum_{n=1}^{\infty} b_n'(t)X_n(x) - \sum b_n(t)X_n''(x) = \sum_{n=1}^{\infty} \gamma_n(t)X_n(x)$. Equating coefficients of $X_n(x)$, we get $b_n'(t) + \lambda_n b_n(t) = \gamma_n(t)$. The initial conditions imply $u(x, 0) = \sum b_n(0)X_n(x) = f(x) \implies b_n(0) = \beta_n$.

iii. [5] Using the integrating factor $e^{\lambda_n t}$ gives

$$\frac{d}{dt}\left[e^{\lambda_n t} b_n(t)\right] = e^{\lambda_n t}\gamma_n(t).$$

So

$$e^{\lambda_n t} b_n(t) - b_n(0) = \int_0^t e^{\lambda_n \tau}\gamma_n(\tau)\, d\tau.$$

So

$$\begin{aligned} b_n(t) &= e^{\lambda_n t} b_n(0) + e^{-\lambda_n t}\int_0^t e^{\lambda_n \tau}\gamma_n(\tau)\, d\tau \\ &= e^{\lambda_n t}\beta_n + \int_0^t e^{-\lambda_n (t-\tau)}\gamma_n(\tau)\, d\tau. \end{aligned}$$

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