MA3603

CITY UNIVERSITY

London

BSc Honours Degree in Mathematical Science
Mathematical Science with Statistics
Mathematical Science with Computer Science
Mathematical Science with Finance and Economics
Mathematics and Finance

Part III

Mathematical Methods II

2013

Time allowed: 3 hours

Full marks may be obtained for correct answers to FOUR of the six questions, with TWO questions selected from each section.

In each section, if more than TWO questions are answered, the best TWO marks will be credited.

Each question carries 25 marks.

SECTION A

1) (*i*) [4 marks] The linear fractional transformation is defined as

$$w = T(z) = \frac{az+b}{cz+d} \qquad \text{for } ad-bc \neq 0; a,b,c,d \in \mathbb{C}.$$

What type of map do we obtain when giving up the restriction $ad-bc \neq 0$? For which choices of the constants a,b,c,d does this map reduce to a) a rotation by $-\pi/6$ followed by a translation by $1+i\sqrt{3}$, b) a translation by $1+i\sqrt{3}$ followed by a rotation by $-\pi/6$, and c) an inversion followed by a scaling of $\mu \in \mathbb{R}$. Report your answers for a,b,c,d in the form $x+iy$ with $x,y \in \mathbb{R}$.

(*ii*) [8 marks] Given the two linear fractional transformations

$$T_1(z) = \frac{z-e^{i\pi/3}}{e^{-i\pi/3}z+1} \qquad \text{and} \qquad T_2(z) = \frac{e^{i\pi 2/3}z-1}{z+e^{i\pi/2}}$$

compute the linear fractional transformation equivalent to the composition

$$T_2 \circ T_1(z).$$

Why is your answer expected to be a linear fractional transformation? Report your answers for a,b,c,d in the form $x+iy$ with $x,y \in \mathbb{R}$. Decompose your result into a succession of rotations, translations and inversions.

(*iii*) [8 marks] Prove that the linear fractional transformation

$$T(z) = e^{i\theta}\frac{z-\gamma}{\bar{\gamma}z-1} \qquad \text{for } \theta \in \mathbb{R}, \gamma \in \mathbb{C},$$

maps a circle of radius one into a circle of radius one. Fix θ and γ in such a way that $T(z)$ leaves the unit circle invariant and maps the line passing through the points $z_1 = 0$, $z_2 = (1-i\sqrt{3})/2$ to the line passing through $w_1 = 1+i\sqrt{3}$, $w_2 = -(1+i\sqrt{3})/2$.

(*iv*) [5 marks] Verify that the function

$$u(x,y) = e^x\left[x\cos(y) - y\sin(y)\right]$$

is a harmonic function. Compute the conjugate harmonic function for $u(x,y)$ and subsequently construct an analytic function with $u(x,y)$ as real part.

2) (*i*) [3 marks] Define what is meant by the exponential growth of a function. Compute the exponential growth for the functions

$$f_1(x) = 5, \qquad \text{and} \qquad f_2(x) = e^{3\beta x} \text{ with } \beta \in \mathbb{R}.$$

(*ii*) [3 marks] Provide the explicit argument of how the exponential growth of a function is used to ensure the existence of a Laplace transform.

(*iii*) [3 marks] Derive a formula that expresses the Laplace transform of the fourth derivative of $u(x)$, i.e., $\mathcal{L}u^{iv}(x)$, in terms of the Laplace transform $\mathcal{L}u(x)$.

(*iv*) [10 marks] Employ Laplace transforms to solve the fourth order differential equation

$$u^{iv}(x) - u(x) = 0$$

with initial conditions $u(0) = 0$, $u'(0) = 1$, $u''(0) = 0$ and $u'''(0) = 0$. You may use information from the following table in this question

$u(x)$	$\mathcal{L}u(x)$
1	$\frac{1}{x}$
x	$\frac{1}{x^2}$
$\cos x$	$\frac{x}{1+x^2}$
$\sin x$	$\frac{1}{1+x^2}$
$x \cos x$	$\frac{x^2-1}{(1+x^2)^2}$
$x \sin x$	$\frac{2x}{(1+x^2)^2}$
$\exp(-x)$	$\frac{1}{1+x}$
$\sinh x$	$\frac{1}{x^2-1}$

(*v*) [6 marks] Use Fourier transforms to solve the Fredholm integral equation for $\phi(x)$

$$e^{-x^2} = \kappa \int_{-\infty}^{\infty} e^{-(x-s/\lambda)^2} \phi(s/\lambda) ds,$$

where $\lambda, \kappa \in \mathbb{R}$.

3) (*i*) [10 marks] Find a conformal map $w = f(z)$ which maps the wedge region in the z-plane

$$\mathcal{W} = \{r, \theta : r \in \mathbb{R}^+, 0 \leq \theta \leq \frac{\pi}{8}\}$$

onto the unit disc $|w| \leq 1$. Verify that the points $z_1 = 0$, $z_2 = 1$, $z_3 = e^{i\pi/8}$ and $z_4 = e^{i\pi/16}$ are mapped correctly. Which theorem guarantees that such map exits? Is this map unique?

(*ii*) [3 marks] Define the branch, the branch cut and the branch point of a multi-valued function.

(*iii*) [12 marks] Find the largest domain on which the function

$$g(z) = \ln\left(\frac{z^2 - 36}{z - 9}\right)$$

is single valued and analytic. Take the principal branch cut for the logarithmic function.

SECTION B

4) (*a*) Consider the initial value problem

$$\frac{dy}{dx} = 3x^2(y+1), \quad x \in \mathbf{R}, \qquad y(0) = 0.$$

(*i*) [1 mark] Write the differential equation as an integral equation $y(x) = L[x, y(x)]$.

(*ii*) [4 marks] Define the Picard iterates by $y_{n+1}(x) = L[x, y_n(x)]$. Taking $y_0(x) = 0$, find $y_1(x)$ and $y_2(x)$.

(*iii*) [5 marks] Use Picard's Theorem to show that the solution of the problem is unique for $0 \le x \le 2/3$.

(*b*) Consider the nonhomogeneous boundary value problem

$$L[y] := \frac{d}{dx}\left[\frac{1}{x^3}\frac{dy}{dx}\right] + \frac{3}{x^5}y = f(x), \qquad 1 < x < 2,$$

$$y(1) = 0, \qquad y(2) - 2y'(2) = 0.$$

(*i*) [4 marks] Show that $y_1(x) = x$ and $y_2(x) = x^3$ form a fundamental set of solutions to the associated homogeneous equation $L[y] = 0$.

(*ii*) [9 marks] State the properties of the Green's function for a self-adjoint equation with homogeneous boundary conditions. Use these properties to derive the Green's function for the problem under consideration. (You may also use the symmetry property.)

(*iii*) [2 marks] Use the Green's function to express the solution of the boundary value problem as a sum of two integrals.

Turn over . . .

5) (*a*) [6 marks] Find the solution of the partial differential equation

$$y(x^2+1)u_x + x(y^2+1)u_y = 0, \qquad x>0, \quad y>0,$$

that satisfies the condition $u(x,0) = x^2$.

(*b*) Consider the partial differential equation

$$yu_{xx} + (x+y)u_{xy} + xu_{yy} = -2, \qquad x \neq y.$$

(*i*) [5 marks] Show that the equation is hyperbolic. Find the equations of its characteristics.

(*ii*) [10 marks] By transforming to the coordinates $z = x^2 - y^2$ and $w = x - y$, show that the canonical form of the equation is

$$w^2 u_{zw} + w u_z = 1.$$

(*iii*) [4 marks] Hence find the general solution $u(x,y)$ of the equation. (Hint: integrate with respect to z first.)

Turn over ...

6) (*a*) Consider the regular Sturm-Liouville problem

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(x)y + \lambda r(x)y = 0, \qquad \alpha < x < \beta,$$

$$y(\alpha) = 0, \quad 2y(\beta) - y'(\beta) = 0,$$

with $p(x) > 0$ and $r(x) > 0$ for $\alpha \le x \le \beta$.

(*i*) [2 mark] State the orthogonality property satisfied by the eigenfunctions.

(*ii*) [4 marks] Prove that the eigenvalues are real.

(*b*) Consider the eigenvalue problem

$$\frac{d^2y}{dx^2} + \lambda y = 0, \qquad 0 < x < 1,$$
$$y(0) = 0, \quad 2y(1) - y'(1) = 0.$$

(*i*) [5 marks] Show that the problem has one negative eigenvalue.

(*ii*) [2 mark] Show that 0 is not an eigenvalue of the problem.

(*iii*) [7 marks] Show that the problem has an infinite number of eigenvalues λ_n given by μ_n^2, where μ_n is a positive solution of the equation $2\tan\mu = \mu$. Show that the least of these eigenvalues lies between π^2 and $\frac{9}{4}\pi^2$. Show that as n increases, the difference $\lambda_{n+1} - \lambda_n$ between consecutive eigenvalues tends to $2(n+1)\pi^2$.

(*iv*) [2 marks] Write down the eigenfunctions of the problem.

(*v*) [3 marks] Consider the eigenvalue problem obtained by replacing the boundary condition $2y(1) - y'(1)$ by $y(1) - 2y'(1)$. How many negative eigenvalues does it have? Justify your answer.

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SECTION A

1) (*i*) Since we have $T'(z) = (ad-bc)/(d+cz)^2$ the map becomes a constant. [4]
We have

a) $$f_T^{1+i\sqrt{3}} \circ f_R^{\exp(-i\pi/6)}(z) = \exp(-i\pi/6)z + 1 + i\sqrt{3} \ ,$$
$$\Rightarrow \ a = e^{-i\pi/6} = \frac{\sqrt{3}}{2} - \frac{i}{2}, \quad b = 1 + i\sqrt{3}, \quad c = 0, \quad d = 1,$$

b) $$f_R^{\exp(-i\pi/6)} \circ f_T^{1+i\sqrt{3}}(z) = \exp(-i\pi/6)z + \exp(-i\pi/6)(1 + i\sqrt{3}),$$
$$\Rightarrow \ a = e^{-i\pi/6} = \frac{\sqrt{3}}{2} - \frac{i}{2}, \quad b = i + \sqrt{3}, \quad c = 0, \quad d = 1,$$

c) $$f_R^{\mu} \circ f_I(z) = \frac{\mu}{z},$$
$$\Rightarrow \ a = 0, \quad b = \mu, \quad c = 1, \quad d = 0.$$

(*ii*) We compute [6]

$$\begin{aligned}
T_2 \circ T_1(z) &= \frac{e^{i\pi 2/3}\frac{z-e^{i\pi/3}}{e^{-i\pi/3}z+1} - 1}{\frac{z-e^{i\pi/3}}{e^{-i\pi/3}z+1} + e^{i\pi/2}} = \frac{e^{i\pi 2/3}(z - e^{i\pi/3}) - (e^{-i\pi/3}z + 1)}{z - e^{i\pi/3} + e^{i\pi/2}(e^{-i\pi/3}z + 1)} \\
&= \frac{(e^{i\pi 2/3} - e^{-i\pi/3})z}{z(1 + e^{i\pi/6}) + i - e^{i\pi/3}} \\
&= \frac{2e^{i\pi 2/3}z}{z(1 + e^{i\pi/6}) + i - e^{i\pi/3}} \\
&= \frac{(i\sqrt{3} - 1)z}{z(1 + \sqrt{3}/2 + i/2) - 1/2 + i(1 - \sqrt{3}/2)} \\
&= \left(\frac{4iz}{\left(\sqrt{3} + (2 - i)\right) z - (1 - i)\left(\sqrt{3} - 1\right)}\right)
\end{aligned}$$

For $c \neq 0$ we have [2]

$$T(z) = f_T^{a/c} \circ f_R^{(bc-ad)/c} \circ f_I \circ f_T^{d} \circ f_R^{c}(z),$$

such that

$$T_2 \circ T_1(z) = f_T^{2e^{i\pi 2/3}/(1+e^{i\pi/6})} \circ f_R^{2e^{i\pi 2/3}(e^{i\pi/3}-i)/(1+e^{i\pi/6})} \circ f_I \circ f_T^{i-e^{i\pi/3}} \circ f_R^{1+e^{i\pi/6}}(z)$$

(*iii*) We need to show that $|T(z)| = 1$ for $|z| = 1$. For $|T(z)| = 1$ we [8] obtain

$$\begin{aligned}(z-\gamma)(\bar z-\bar\gamma) &= (\bar\gamma z-1)(\gamma\bar z-1)\\ |z|^2-z\bar\gamma-\gamma\bar z+|\gamma|^2 &= |z|^2|\gamma|^2-z\bar\gamma-\gamma\bar z+1\end{aligned}$$

which is an identity for $|z|=1$. Parameterizing $\gamma = re^{i\phi}$ we compute

$$\begin{aligned}T(0) &= e^{i\theta}\gamma = re^{i(\theta+\phi)} = 1+i\sqrt{3} = 2e^{i\pi/3}\\ &\implies r=2, \qquad \theta = \pi/3-\phi\end{aligned}$$

Next compute

$$T\left(\frac{1-i\sqrt{3}}{2}\right) = T\left(e^{-i\pi/3}\right) = \frac{e^{i\left(\frac{\pi}{3}-\phi\right)}\left(e^{-\frac{i\pi}{3}}-2e^{i\phi}\right)}{-1+2e^{-i\phi-\frac{i\pi}{3}}} = -e^{i\pi/3}$$

Solve this for ϕ, for instance

$$\frac{e^{-i\phi}-2e^{i\pi/3}}{-1+2e^{-i\phi-\frac{i\pi}{3}}} = -e^{i\pi/3} \quad \Rightarrow e^{-i\phi} = e^{i\pi/3} \;\Rightarrow \phi = -\pi/3$$

Therefore $\underline{\theta = 2\pi/3}$ and $\underline{\gamma = 2e^{-i\pi/3}}$.

(iv) We compute 5

$$\begin{aligned}\partial_x u(x,y) &= e^x\cos y + e^x(x\cos y - y\sin y)\\ \partial_x^2 u(x,y) &= 2e^x\cos y + e^x(x\cos y - y\sin y)\\ \partial_y u(x,y) &= -e^x(x\sin y+\sin y + y\cos y)\\ \partial_y^2 u(x,y) &= e^x(-x\cos y + y\sin y - 2\cos y)\end{aligned}$$

such that $\Delta u(x,y) = \partial_x^2 u(x,y) + \partial_y^2 u(x,y) = 0$. This means $u(x,y)$ is a harmonic functions. The conjugate harmonic function $v(x,y)$ is obtained from solving the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

$$\begin{aligned}\frac{\partial u}{\partial x} &= e^x\cos y + e^x(x\cos y - y\sin y) = \frac{\partial v}{\partial y}\\ &\Rightarrow\; v = e^x\int(\cos y - y\sin y)dy + e^x x\int\cos y dy\\ &= e^x y\cos y + e^x f(x) + xe^x\sin y + xe^x g(x)\\ \frac{\partial u}{\partial y} &= -e^x(x\sin y + \sin y + y\cos y)\\ &\Rightarrow\; v = \sin y\int xe^x dx + (\sin y + y\cos y)\int e^x dx\\ &= \sin y(x-1)e^x + \sin y\tilde f(y) + (e^x+\tilde g(y))(\sin y + y\cos y)\\ &= xe^x\sin y + \sin y\tilde f(y) + e^x y\cos y + \tilde g(y)(\sin y + y\cos y)\end{aligned}$$

Thus $f(x) = g(x) = \tilde{f}(y) = \tilde{g}(y) = 0$ and

$$v(x,y) = e^x y \cos y + x e^x \sin y.$$

An analytic function with $u(x,y)$ as real part is therefore

$$f(x,y) = e^x (x \cos y - y \sin y) + i e^x (y \cos y + x \sin y).$$

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2) (i) *The function* u(x) *is said to have* exponential growth α *if there exists a constant* μ *such that*

$$|u(x)| \leq \mu e^{\alpha x} \qquad \text{for } x > 0, \text{ with } \alpha, \mu \in \mathbb{R}.$$

We compute

$$\begin{aligned} |f_1(x)| &= |5| \leq 6 && \Rightarrow \alpha = 0 \\ |f_2(x)| &= \left|2e^{3\beta x}\right| \leq 2e^{\alpha x} && \Rightarrow \alpha > 3\beta \end{aligned}$$

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(ii) The existence of the Laplace transform is guaranteed by the following argument

$$\begin{aligned} \mathcal{L}u(x) &\leq \int_0^\infty |u(t)|\, e^{-tx} dt \\ &\leq \mu \int_0^\infty e^{\alpha x} e^{-tx} dt \\ &= \mu \int_0^\infty e^{(\alpha - x)t} dt < \infty \qquad \text{for } x > \alpha. \end{aligned}$$

(iii) The Laplace transform for the derivative $u'(x)$ of the function $u(x)$ is

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$$\begin{aligned} \mathcal{L}u'(x) &= \int_0^\infty u'(t) e^{-tx} dt = u(t) e^{-tx}\big|_0^\infty + x \int_0^\infty u(t) e^{-tx} dt \\ &= x\mathcal{L}u(x) - u(0), \end{aligned}$$

where we integrated by parts. Now compute

$$\begin{aligned} \mathcal{L}u^{iv}(x) &= x\mathcal{L}u'''(x) - u'''(0) = x^2 \mathcal{L}u''(x) - xu''(0) - u'''(0) \\ &= x^3 \mathcal{L}u'(x) - x^2 u'(0) - xu''(0) - u'''(0) \\ &= x^4 \mathcal{L}u(x) - x^3 u(0) - x^2 u'(0) - xu''(0) - u'''(0) \end{aligned}$$

(iv) We act on the original equation with the Laplace transform

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$$\mathcal{L}u^{iv}(x) - \mathcal{L}u(x) = 0$$

Using the formula derived in (iii) we rewrite the equation as

$$x^4 \mathcal{L}u(x) - x^3 u(0) - x^2 u'(0) - x u''(0) - u'''(0) - y(x) = 0.$$

With the initial conditions $u(0) = 0$, $u'(0) = 1$, $u''(0) = 0$ and $u'''(0) = 0$ this becomes

$$x^4 \mathcal{L}u(x) - x^2 - \mathcal{L}u(x) = 0.$$

Therefore

$$\mathcal{L}u(x) = \frac{x^2}{x^4 - 1} = \frac{Ax + B}{x^2 - 1} + \frac{Cx + D}{x^2 + 1},$$

where we used a partial fraction expansion. Therefore

$$x^2 = (Ax + B)\left(x^2 + 1\right) + (Cx + D)\left(x^2 - 1\right).$$

For $x = 1$, $x = -1$ and $x = 0$ this becomes

$$1 = 2(A + B), \quad 1 = 2(-A + B) \qquad \text{and} \qquad 0 = B - D.$$

Therefore $A = 0$, $B = 1/2$, $D = 1/2$. The cubic term in x gives $A + C = 0$, such that $C = 0$. Thus

$$\mathcal{L}u(x) = \frac{1/2}{x^2 - 1} + \frac{1/2}{x^2 + 1}.$$

Using the table for the inverse Laplace transform we obtain

$$u(x) = \frac{1}{2}\mathcal{L}^{-1}\left(\frac{1}{x^2 - 1}\right) + \frac{1}{2}\mathcal{L}^{-1}\left(\frac{1}{x^2 + 1}\right) = \frac{1}{2}(\sin x + \sinh x).$$

(v) First scale $s \to \lambda s$, such that the equation becomes **6**

$$e^{-x^2} = \lambda\kappa \int_{-\infty}^{\infty} e^{-(x-s)^2} \phi(s) ds$$

Introducing the function $v(x) = e^{-x^2}$, this can be rewritten as

$$v(x) = \lambda\kappa v * \phi(x)$$

Acting on this equation with the Fourier operator $\mathcal{F}$ gives

$$\mathcal{F}v(x) = \mathcal{F}(\lambda\kappa v * \phi)(x) = \lambda\kappa \mathcal{F}(v * \phi)(x) = \lambda\kappa \mathcal{F}v(x)\mathcal{F}\phi(x).$$

Therefore we obtain

$$\frac{1}{\lambda\kappa} = \mathcal{F}\phi(x),$$

such that

$$\phi(x) = \mathcal{F}^{-1}\frac{1}{\lambda\kappa}(x) = \frac{1}{\lambda\kappa}\mathcal{F}^{-1}1(x) = \frac{1}{\lambda\kappa}\delta(x).$$

3) (*i*) First rotate the wedge region $\mathcal{W}$ by $-i\pi/18$ [6]

$$\hat{w} = \hat{f}(z) = ze^{-i\pi/16},$$

such that the new wedge region is

$$\mathcal{W}' = \{r, \theta : r \in \mathbb{R}^+, -\frac{\pi}{16} \leq \theta < \frac{\pi}{16}\}.$$

Next map this wedge to the entire right half plane by

$$\tilde{w} = \tilde{f}(\hat{w}) = \hat{w}^8.$$

Finally we map the right half plane to the unit disk

$$w = \check{f}(\tilde{w}) = \frac{\tilde{w} - 1}{\tilde{w} + 1}.$$

Thus the map which maps $\mathcal{W}$ onto the unit disk is

$$w = f(z) = \check{f} \circ \tilde{f} \circ \hat{f}(z) = \check{f} \circ \tilde{f}(ze^{-i\pi/16}) = \check{f}(-iz^8) = \frac{z^8 - i}{z^8 + i}.$$

We compute: [2]

$$f(0) = -1, \quad f(1) = -i, \quad f(e^{i\pi/8}) = i, \quad f(e^{i\pi/16}) = 0.$$

The points z_1, z_2, z_3 are on the boundary of $\mathcal{W}$ and are mapped correctly onto the unit circle. The point z_4 is in the interior of the $\mathcal{W}$ and is mapped correctly into the interior of the unit circle.

The existence of the map is guaranteed by the Riemann mapping theorem: *Given a simply connected region $D \subset \mathbb{C}$ (i.e. D has no holes) which is not the entire plane and a point $z_0 \in D$. Then there exists an analytic function $f : z \mapsto w$ which maps D one-to-one onto the interior of the unit disk $|w| < 1$. The uniqueness of the map can be achieved with a suitable normalization condition $f(z_0) = 0$ and $f'(z_0) > 0$.* [2]

[3]

(*ii*) **Definition:** *A branch F(z) of a multi-valued function $f(z)$ is any single valued function, which is analytic in some domain $D \subset \mathbb{C}$, where F(z_0) =$f(z_0)$ for all $z_0 \in D$.*

Definition: *A branch cut is a curve in the complex plane across which an analytic multivalued function is discontinuous.*

Definition: *A point which is shared by all branches of the function is called a branch point.*

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(*iii*) The function $g(z)$ has three branch points at $z = 9$ and at $z = \pm 6$. For the arguments of the logarithm we can write

$$z \pm 6 = |z \pm 6|\, e^{i\theta_{1/2}} \qquad \text{and} \qquad z - 9 = |z - 9|\, e^{i\theta_3}$$

such that

$$\begin{aligned} g(z) &= \ln\left(\frac{z^2 - 36}{z - 9}\right) = \ln(z+6) + \ln(z-6) - \ln(z-9) \\ &= \ln\left|\frac{z^2 - 36}{z - 9}\right| + i(\theta_1 + \theta_2 - \theta_3) \end{aligned}$$

We have now various choices for the restriction on θ_1, θ_2 and θ_3 :
Assuming the principal values for the logarithms means:

$$-\pi < \theta_1, \theta_2, \theta_3 \leq \pi$$

Let us now consider the different regions on the real axis:

- $z \in (9, \infty)$: On this part of the axis there is no problem as θ_1, θ_2 and θ_3 are all continuous when crossing the axis.
- $z \in (6, 9)$: On this line segment θ_1 and θ_2 are continuous, but θ_3 jumps and therefore we require a cut.
- $z \in (-6, 6)$: When crossing this part of the axis both θ_2 and θ_3 are discontinuous. However, the relevant quantity, which is the difference $\theta_1 + \theta_2 - \theta_3$ is continuous. Above the axis we have $\theta_1 = 0$, $\theta_2 = \theta_3 = \pi$, such that $\theta_1 + \theta_2 - \theta_3 = 0$ and below the axis we have $\theta_1 = 0$, $\theta_2 = \theta_3 = -\pi$ and therefore also $\theta_1 + \theta_2 - \theta_3 = 0$. This means no cut is required on this segment.
- $z \in (-\infty, -6)$: On this line segment we have above the axis $\theta_1 = \theta_2 = \theta_3 = \pi$ such that $\theta_1 + \theta_2 - \theta_3 = \pi$ and below the axis we have $\theta_1 = \theta_2 = \theta_3 = -\pi$ such that $\theta_1 + \theta_2 - \theta_3 = -\pi$. This means the function is discontinuous and we need a branch cut to make it analytic.

Overall we therefore require branch cuts at the line segment $(-\infty, -6)$ and $(6, 9)$ in order to make the function $g(z)$ single valued and analytic.

4) (a) (i) (1 mark) Integrating the differential equation, we have $y = L[x, y(x)]$, where

$$L[x, y(x)] = \int_0^x 3s^2(y(s) + 1)ds.$$

(ii) (4 marks) The Picard iterates are defined recursively as

$$y_{n+1}(x) = \int_0^x 3s^2(y_n(s)+1)ds.$$

So if $y_1(x) = 0$, we have

$$y_1(x) = \int_0^x 3s^2(0+1)ds = x^3,$$

and

$$y_2(x) = \int_0^x 3s^2(s^3+1)ds = \frac{1}{2}x^6 + x^3.$$

(iii) (5 marks) The DE has the form $dy/dx = f(x,y)$, where $f(x,y) = 3x^2(y+1)$. Both f and $f_y = 3x^2$ are continuous for all x and y, and thus continuous in the rectangle

$$R = \{(x,y) \mid 0 \le x \le a, \quad -b \le y \le b\}$$

for all $a, b > 0$. So the hypotheses of Picard's are satisfied. We have $M = \max_R |f(x,y)| = 3a^2(b+1)$, and the theorem states that the initial value problem has a unique solution for $0 \le x \le \alpha$, where $\alpha = \min\left\{a, \frac{b}{M}\right\} = \min\left\{a, \frac{b}{3a^2(b+1)}\right\}$. Taking $a = 2/3$ and $b = 8$, we obtain $\alpha = 2/3$. (Other choices of a and b will serve.)

(b) (i) (4 marks) Calculate $L[x] = -\frac{3}{x^4} + \frac{3}{x^4} = 0$ and $L[x^3] = -\frac{3}{x^2} + \frac{3}{x^2} = 0$. The Wronskian $W(y_1, y_2)(x) = \begin{vmatrix} x & x^3 \\ 1 & 3x^2 \end{vmatrix} = 2x^3 \neq 0$ for $x > 1$.

So f_1 and f_2 form a fundamental set of solutions.

(ii) (9 marks) The Green's function for the self-adjoint equation $(py')' + qy = -f$ has the following properties:

(1) $G(x,t)$ satisfies the homogeneous DE with respect to x ($x \neq t$)

(2) $G(x,t)$ satisfies the boundary conditions

(3) $G(x,t)$ is continuous at $x = t$

(4) $\frac{\partial G}{\partial x}$ is discontinuous at $x = t$ with $\frac{\partial G}{\partial x}|_{x=t+0} - \frac{\partial G}{\partial x}|_{x=t-0} = -\frac{1}{p(t)}$

By the first property

$$G(x,t) = \begin{cases} a(t)x + b(t)x^3, & 1 \le x \le t \\ c(t)x + d(t)x^3, & t \le x \le 2. \end{cases}$$

The second property implies $G(1,t) = a + b = 0 \implies a = -b$, and $G(2,t) - 2G_x(2,t) = (2c + 8d) - 2(c + 12d) = 0 \implies d = 0$. Thus

$$G(x,t) = \begin{cases} b(t)(x^3 - x), & 1 \leq x \leq t \\ c(t)x, & t \leq x \leq 2. \end{cases}$$

Next the symmetry property implies

$$G(x,t) = \begin{cases} kt(x^3 - x), & 1 \leq x \leq t \\ k(t^3 - t)x, & t \leq x \leq 2 \end{cases}$$

for some constant k, and then property (3) is automatically satisfied. Finally using the last property, we get $k(t^3-t) - kt(3t^2-1) = -t^3 \implies k = \frac{1}{2}$. Hence

$$G(x,t) = \begin{cases} \frac{1}{2}t(x^3 - x), & 1 \leq x \leq t \\ \frac{1}{2}(t^3 - t)x, & t \leq x \leq 2. \end{cases}$$

(iii) (2 marks) So the solution is

$$y(x) = \int_1^2 G(x,t)(-f(t))\,dt = -\frac{x}{2}\int_1^x (t^3-t)f(t)\,dt - \frac{1}{2}(x^3-x)\int_x^2 tf(t)\,dt.$$

5) (a) (6 marks) The characteristic equation is

$$\frac{dy}{dx} = \frac{x(y^2+1)}{y(x^2+1)}.$$

Separating variables and integrating, we obtain

$$\int \frac{y\,dy}{y^2+1} = \int \frac{x\,dx}{x^2+1},$$

so that

$$\ln(y^2+1) = \ln(x^2+1) + C.$$

Exponentiating, we obtain the characteristic curves

$$\frac{y^2+1}{x^2+1} = K.$$

Hence the general solution of the PDE is given by

$$u(x,y) = f(K) = f\left(\frac{y^2+1}{x^2+1}\right),$$

where f is an arbitrary function. The initial condition gives

$$x^2 = u(x,0) = f\left(\frac{1}{x^2+1}\right),$$

and therefore $f(z) = \frac{1}{z} - 1$. Hence the particular solution is

$$u(x,y) = \frac{x^2+1}{y^2+1} - 1 = \frac{x^2-y^2}{y^2+1}.$$

(b) (i) (5 marks) We have $b^2 - 4ac = (x+y)^2 - 4yx = (x-y)^2 > 0$, since $x \neq y$. So the equation is hyperbolic. The characteristics are solutions of the differential equation

$$y\left(\frac{dy}{dx}\right)^2 - (x+y)\frac{dy}{dx} + x = 0.$$

So

$$dy/dx = \frac{1}{2y}\left(x + y \pm (x-y)\right) = \frac{x}{y}, 1.$$

giving characteristics $y^2 = x^2 + c_1$ and $y = x + c_2$.

(ii) (10 marks) The characteristic coordinates can be taken to be $z = -c_1 = x^2 - y^2$ and $w = -c_2 = x - y$. To transform to canonical form, we calculate

$$\begin{aligned}
u_x &= u_z z_x + u_w w_x = 2xu_z + u_w. \\
u_y &= u_z z_y + u_w w_y = -2yu_z - u_w. \\
u_{xx} &= 2u_z + 2x(u_z)_x + (u_w)_x \\
&= 2u_z + 2x(2xu_{zz} + u_{zw}) + 2xu_{wz} + u_{ww} \\
&= 4x^2u_{zz} + 4xu_{zw} + u_{ww} + 2u_z. \\
u_{yy} &= -2u_z - 2y(u_z)_y - (u_w)_y \\
&= -2u_z - 2y(-2yu_{zz} - u_{zw}) + 2yu_{wz} + u_{ww} \\
&= 4y^2u_{zz} + 4yu_{zw} + u_{ww} - 2u_z. \\
u_{xy} &= 2x(u_z)_y + (u_w)_y \\
&= 2x(-2yu_{zz} - u_{zw}) - 2yu_{wz} - u_{ww} \\
&= -4xyu_{zz} - 2(x+y)u_{wz} - u_{ww}
\end{aligned}$$

So the lefthand side of the PDE transforms to

$$\left(y(4x^2) + (x+y)(-4xy) + x(4y^2)\right)u_{zz}$$

$$+\left(y(4x) + (x+y)(-2)(x+y) + x(4y)\right)u_{zw}$$

$$+\left(y(1)+(x+y)(-1)+x(1)\right)u_{ww}$$
$$+\left(y(2)+(x+y)(0)+x(-2)\right)u_z$$

which is equal to

$$-2(x-y)^2u_{zw}-2(x-y)u_z.$$

So dividing the original differential equation by -2, we obtain

$$w^2u_{zw}+wu_z=1.$$

(iii) (4 marks) Integrating the DE with respect to z we have

$$w^2u_w+wu=z+f_1(w),$$

a first-order linear equation in w. Dividing by w we obtain

$$\frac{d}{dw}(wu)=\frac{z}{w}+f_2(w),$$

and then, integrating,

$$wu=z\ln(w)+f_3(w)+g(z).$$

So

$$\begin{aligned}u&=\frac{z\ln w}{w}+\frac{g(z)}{w}+f(w)\\&=(x+y)\ln(x-y)+\frac{g(x^2-y^2)}{x-y}+f(x-y)\end{aligned}$$

where f and g are arbitrary functions.

6) (a) (i) (2 marks)

$$\int_\alpha^\beta r(x)\phi(x)\psi(x)\,dx=0$$

where $\phi(x)$ and $\psi(x)$ are eigenfunctions corresponding to distinct eigenvalues.

(ii) (4 marks) Suppose λ (possibly complex) is an eigenvalue with associated eigenfunction $\phi(x)$, so

$$(p\phi')'+\lambda r\phi=0.$$

Taking the complex conjugate gives

$$(p\overline{\phi}')'+\overline{\lambda}r\overline{\phi}=0.$$

So $\overline{\lambda}$ is also an eigenvalue with associated eigenfunction $\overline{\phi}(x)$. If $\lambda \neq \overline{\lambda}$, then by the orthogonality property

$$\int_\alpha^\beta r(x)\phi(x)\overline{\phi}(x)\,dx = \int_\alpha^\beta r(x)\,|\phi(x)|^2\,dx = 0.$$

Since $r(x) > 0$, the integrand is positive and the integral cannot be zero. Hence $\lambda = \overline{\lambda}$ so that λ is real.

(b) (i) (5 marks) Suppose $\lambda < 0$, and put $\lambda = -\mu^2$, where μ is positive, real. Then the DE becomes $y''^2 y = 0$, which has general solution $y(x) = A\cosh\mu x + B\sinh\mu x$. The first BC implies $A = 0$, and then the second that $2y(1) - y'(1) = -\mu B\cosh x + 2B\sinh x = 0 \implies \tanh x = \frac{1}{2}\mu$. This has one solution for positive μ, as a plot of the graphs of $z = \tanh\mu$ and $z = \frac{1}{2}\mu$ shows.

(ii) (2 marks) When $\lambda = 0$, the differential equation $y'' = 0$ has general solution $y = Ax + B$. The boundary conditions are $y(0) = B = 0$ and $2y(1) - y'(1) = A = 0$, so there are no non-trivial solutions. Therefore 0 is not an eigenvalue.

(iii) (7 marks) Since $\lambda > 0$, let $\lambda = \mu^2$ where μ is positive, real. Then the DE is $y''^2 y = 0$, which has GS $y(x) = A\cos\mu x + B\sin\mu x$. The first BC gives $A = 0$, and the second $2y(1) - y'(1) = 4B\sin\mu + B\cos\mu = 0$. When $B \neq 0$, this implies $2\tan\mu - \mu = 0$.
Consider the curves $z = \tan\mu$ and $z = \frac{1}{2}\mu$. The points of intersection correspond to the eigenvalues $\lambda = \mu^2$.The function $z = \tan\mu$ has positive zeros at multiples of π and vertical asymptotes $\mu = \pi/2, 3\pi/2$, approached from the left as $z \to \infty$, and so there are points of intersection of the two curves between every pair of consecutive positive asymptotes, and we label them $\mu_1 < \mu_2 < \ldots$. Since $\tan'(0) = 1 > \frac{1}{2}$, note the the first point of intersection for $\mu > 0$ is between the first positive zero and the second positive asymptote, which occur at $\mu = \pi$ and $\mu = \frac{3}{2}\pi$ respectively. Hence λ_1 is between the squares π^2 and $\frac{9}{4}\pi^2$. As n grows large μ_n approaches $(2n+1)\pi/2$. Thus $\lambda_{n+1} - \lambda_n$ tends to $(2n+3)^2\pi^2/4 - (2n+1)^2\pi^2/4 = 2(n+1)\pi^2$.

(iv) (2 marks) From (i), $\psi(x) = \sinh(\nu x)$, where ν is the unique positive solution of $\tanh\nu = \frac{1}{2}\nu$, and, from (iii), $\phi_n(x) = \sin(\mu_n x)$, where μ_n, $n = 1, 2, \ldots$ are the positive solutions of $\tan\mu = \frac{1}{2}\mu$.

(v) (3 marks) It has no negative eigenvalues. To see that, follow the argument in (i), obtaining the general solution $y(x) = A\cosh\mu x + B\sinh\mu x$. The first BC still implies $A = 0$, and then the (modified) second condition implies $y(1) - 2y'(1) = -\mu 2B\cosh\mu + B\sinh\mu = 0 \implies \tanh\mu = 2\mu$. This has no solution for positive

μ, as $f(\mu) = 2\mu - \tanh\mu$ is an increasing function vanishing at $\mu = 0$; indeed $f'(\mu) = 2 - \operatorname{sech}^2\mu = 1 + \tanh^2\mu > 0$.

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